10.8 Power Series

DEFINITION: If $c_0, c_1, c_2, \dots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k(x-x_0)^k = c_0 + c_1(x-x_0) + c_2(x-x_0)^2 + \dots + c_k(x-x_0)^k + \dots$$

is called a **power series in $\boldsymbol{x - x_0}$.**

EXAMPLE:

1. $\displaystyle\sum_{k=0}^{\infty} \frac{(x-1)^k}{k!} = 1 + (x-1) + \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots$

2. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{(x+2)^k}{k!} = 1 - (x+2) + \frac{(x+2)^2}{2!} - \frac{(x+2)^3}{3!} + \dots$

THEOREM: For a power series $\sum c_k(x-x_0)^k$, exactly one of the following is true:

(a) The series converges only for $x = x_0$.

(b) The series converges absolutely (and hence converges) for all real values of x.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$, and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

EXAMPLE: Find the interval of convergence of the series $\displaystyle\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2}$.

SOLUTION: By the ratio test for absolute convergence we have

$$\begin{aligned}
\rho = \lim_{k\to\infty} \left| \frac{u_{k+1}}{u_k} \right| &= \lim_{k\to\infty} \left| \frac{(x-5)^{k+1}}{(k+1)^2} \cdot \frac{k^2}{(x-5)^k} \right| \\
&= \lim_{k\to\infty} \left[|x-5| \left(\frac{k}{k+1} \right)^2 \right] = |x-5| \lim_{k\to\infty} \left(\frac{k}{k+1} \right)^2 = |x-5|
\end{aligned}$$

Thus, the series converges absolutely if $|x-5| < 1$, or $-1 < x-5 < 1$, or $4 < x < 6$. The series diverges if $x < 4$ or $x > 6$.

Now we examine the endpoints. If $x = 6$, then

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(6-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{1}{k^2}$$

which converges by the p-series test. Similarly, if $x = 4$, then

$$\sum_{k=1}^{\infty} \frac{(x-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(4-5)^k}{k^2} = \sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$$

which converges by the alternating series test. So, the interval of convergence is $[4, 6]$ and the radius of convergence is $R = 1$.