10.8 Power Series

DEFINITION: If $c_0, c_1, c_2, \ldots$ are constants and x is a variable, then a series of the form

$$\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + \ldots + c_k x^k + \ldots$$

is called a **power series in x.**

EXAMPLE:

1. $\displaystyle\sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \ldots$

2. $\displaystyle\sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots$

3. $\displaystyle\sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots$

THEOREM: For any power series in x, exactly one of the following is true:

(a) The series converges only for $x = 0$.

(b) The series converges absolutely (and hence converges) for all real values of x.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(-R, R)$, and diverges if $x < -R$ or $x > R$. At either of the values $x = -R$ or $x = R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.

THEOREM: For a power series $\sum c_k (x - x_0)^k$, exactly one of the following is true:

(a) The series converges only for $x = x_0$.

(b) The series converges absolutely (and hence converges) for all real values of x.

(c) The series converges absolutely (and hence converges) for all x in some finite open interval $(x_0 - R, x_0 + R)$, and diverges if $x < x_0 - R$ or $x > x_0 + R$. At either of the values $x = x_0 - R$ or $x = x_0 + R$, the series may converge absolutely, converge conditionally, or diverge, depending on the particular series.