10.6 The Comparison, Ratio, and Root Tests

THEOREM (**The Ratio Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}.$$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the ratio test to determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{k+1}{k!}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{3^{k+1}}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$ (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k)!}{3^k}$ (e) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{3k+4}$

SOLUTION:

(a) We have $\rho = \lim\limits_{k\to+\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \dfrac{k+2}{(k+1)!} \cdot \dfrac{k!}{k+1} = \lim\limits_{k\to+\infty} \dfrac{k+2}{(k+1)(k+1)} = 0 < 1$, therefore the series converges.

(b) We have $\rho = \lim\limits_{k\to+\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \dfrac{k+1}{3^{k+2}} \cdot \dfrac{3^{k+1}}{k} = \lim\limits_{k\to+\infty} \dfrac{k+1}{3k} = \dfrac{1}{3} < 1$, therefore the series converges.

(c) We have $\rho = \lim\limits_{k\to+\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \dfrac{(2k+2)^{k+3}}{(k+2)!} \cdot \dfrac{(k+1)!}{(2k)^{k+2}} = \lim\limits_{k\to+\infty} \dfrac{(2k+2)^{k+2}(2k+2)}{(k+2)!} \cdot \dfrac{(k+1)!}{(2k)^{k+2}} = \lim\limits_{k\to+\infty} \dfrac{(2k+2)^{k+2}(2k+2)(k+1)!}{(2k)^{k+2}(k+2)!} = \lim\limits_{k\to+\infty} \left(1+\dfrac{1}{k}\right)^{k+2} \dfrac{2k+2}{k+2} = 2e > 1$, therefore the series diverges.

(d) We have $\rho = \lim\limits_{k\to+\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \dfrac{(2k+2)!}{3^{k+1}} \cdot \dfrac{3^k}{(2k)!} = \lim\limits_{k\to+\infty} \dfrac{(2k+1)(2k+2)}{3} = +\infty$, therefore the series diverges.

(e) We have $\rho = \lim\limits_{k\to+\infty} \dfrac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \dfrac{3k+4}{3k+7} = 1$, therefore the series may converge or diverge, so that another test must be tried. For example, by the integral test, $\displaystyle\int_1^{+\infty} \frac{dx}{3x+4} = \lim_{\ell\to+\infty} \int_1^{\ell} \frac{dx}{3x+4} = \lim_{\ell\to+\infty} \frac{1}{3}\ln(3x+4)\Big]_1^{\ell} = +\infty$, therefore the series diverges.

THEOREM (**The Root Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the root test to determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$

SOLUTION:

(a) We have $\rho = \lim\limits_{k\to+\infty} \sqrt[k]{u_k} = \lim\limits_{k\to+\infty} \dfrac{4k-5}{2k+1} = 2 > 1$, therefore the series diverges.

(b) We have $\rho = \lim\limits_{k\to+\infty} \sqrt[k]{u_k} = \lim\limits_{k\to+\infty} \dfrac{1}{\ln(k+1)} = 0 < 1$, therefore the series converges.

10.7 Alternating Series; Conditional Convergence

DEFINITION: An **alternating series** is a series whose terms alternate between positive and negative.

REMARK: In general, an alternating series has one of the following two forms:

$$\sum_{k=1}^{\infty}(-1)^{k+1}a_k = a_1 - a_2 + a_3 - a_4 + \dots$$

$$\sum_{k=1}^{\infty}(-1)^{k}a_k = -a_1 + a_2 - a_3 + a_4 - \dots$$

THEOREM (**Alternating Series Test**): An alternating series converges if the following two conditions are satisfied:

(a) $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_k \geq \dots$

(b) $\lim\limits_{k\to+\infty} a_k = 0$

DEFINITION: A series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots$$

is said to **converge absolutely** if the series of absolute values

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots + |u_k| + \dots$$

converges and is said to **diverge absolutely** if the series of absolute values diverges.

THEOREM: If the series

$$\sum_{k=1}^{\infty} |u_k| = |u_1| + |u_2| + \dots + |u_k| + \dots$$

converges, then so does the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + \dots + u_k + \dots$$

THEOREM (**Ratio Test for Absolute Convergence**): Let $\sum u_k$ be a series with nonzero terms and suppose that

$$\rho = \lim_{k\to+\infty} \frac{|u_{k+1}|}{|u_k|}$$

(a) If $\rho < 1$, then the series $\sum u_k$ converges absolutely and therefore converges.

(b) If $\rho > 1$ or if $\rho = +\infty$, then the series $\sum u_k$ diverges.

(c) If $\rho = 1$, no conclusion about convergence or absolute convergence can be drawn from this test.