THEOREM (**The Comparison Test**): Let $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that

$$a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3, \ldots, a_k \le b_k, \ldots$$

(a) If the "bigger series" $\sum_{k=1}^{\infty} b_k$ converges, then the "smaller series" $\sum_{k=1}^{\infty} a_k$ also converges.

(b) If the "smaller series" $\sum_{k=1}^{\infty} a_k$ diverges, then the "bigger series" $\sum_{k=1}^{\infty} b_k$ also diverges.

EXAMPLE: Use the comparison test to determine whether the series $\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}-1}$ converges or diverges.

EXAMPLE: Use the comparison test to determine whether the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}-1}$ converges or diverges.

SOLUTION:

Since $\displaystyle\frac{1}{\sqrt[3]{k}-1} > \frac{1}{\sqrt[3]{k}}$ and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}}$ diverges, it follows that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}-1}$ also diverges.

EXAMPLE: Use the comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$ converges or diverges.

EXAMPLE: Use the comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$ converges or diverges.

SOLUTION:

Since $\frac{1}{k^2+k+1} < \frac{1}{k^2}$ and $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, it follows that $\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$ also converges.

EXAMPLE: Use the comparison test to determine whether the series $\sum_{k=2}^{\infty} \frac{1}{k^2 - 1}$ converges or diverges.

EXAMPLE: Use the comparison test to determine whether the series $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ converges or diverges.

SOLUTION:

Since $\frac{1}{k^2-1} < \frac{2}{k^2}$ and $\sum_{k=2}^{\infty} \frac{2}{k^2} = 2\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, it follows that $\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ also converges.

THEOREM (**The Limit Comparison Test**): Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k}.$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

EXAMPLE: Use the limit comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ converge or diverge.

EXAMPLE: Use the limit comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ converge or diverge.

SOLUTION:
Put $a_n = \frac{1}{\sqrt{k}+1}$, $b_n = \frac{1}{\sqrt{k}}$. Then

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k}+1}$$

$$= \lim_{k\to+\infty} \frac{1}{1+\frac{1}{\sqrt{k}}} = 1.$$

Since $\rho = 1$ and $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, it follows that $\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ also diverges.

EXAMPLE: Use the limit comparison test to determine whether the series $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k^2 - k - 1}$ converges or diverges.

EXAMPLE: Use the limit comparison test to determine whether the series $\sum_{k=2}^{\infty} \frac{1}{2k^2 - k - 1}$ converges or diverges.

SOLUTION:

Put $a_n = \frac{1}{2k^2 - k - 1}$, $b_n = \frac{1}{2k^2}$. Then

$$\rho = \lim_{k \to +\infty} \frac{a_k}{b_k} = \lim_{k \to +\infty} \frac{2k^2}{2k^2 - k - 1}$$

$$= \lim_{k \to +\infty} \frac{1}{1 - \frac{1}{2k} - \frac{1}{2k^2}} = 1.$$

Since $\rho = 1$ and $\sum_{k=2}^{\infty} \frac{1}{2k^2} = \frac{1}{2} \sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, it follows that $\sum_{k=2}^{\infty} \frac{1}{2k^2 - k - 1}$ also converges.

EXAMPLE: Use the limit comparison test to determine whether the series $\sum_{k=1}^{\infty} \frac{4k^2 - k + 5}{k^5 + k^4 + 2k - 2}$ converges or diverges.

EXAMPLE: Use the limit comparison test to determine whether the series $\sum_{k=1}^{\infty}\frac{4k^2-k+5}{k^5+k^4+2k-2}$ converges or diverges.

SOLUTION:

Put $a_n=\frac{4k^2-k+5}{k^5+k^4+2k-2}$, $b_n=\frac{4}{k^3}$. Then

$$\rho=\lim_{k\to+\infty}\frac{a_k}{b_k}=\lim_{k\to+\infty}\frac{k^3(4k^2-k+5)}{4(k^5+k^4+2k-2)}$$

$$=\lim_{k\to+\infty}\frac{4k^5-k^4+5k^3}{4k^5+4k^4+8k-8}$$

$$=\lim_{k\to+\infty}\frac{1-\frac{1}{4k}+\frac{5}{4k^2}}{1+\frac{1}{k}+\frac{2}{k^4}-\frac{2}{k^5}}=1.$$

Since $\rho=1$ and $\sum_{k=1}^{\infty}\frac{4}{k^3}=4\sum_{k=1}^{\infty}\frac{1}{k^3}$ converges, it follows that $\sum_{k=1}^{\infty}\frac{4k^2-k+5}{k^5+k^4+2k-2}$ also converges.

THEOREM (**The Ratio Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}.$$

(a) If $\rho < 1$, the series converges.

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{k+1}{k!}$ converges or diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{k+1}{k!}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{k+2}{(k+1)!} \cdot \frac{k!}{k+1}$$

$$= \lim_{k\to+\infty} \frac{k+2}{(k+1)(k+1)} = 0 < 1,$$

therefore the series converges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{k}{3^{k+1}}$ converges or diverges.

EXAMPLE: Use the ratio test to determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{k}{3^{k+1}}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{k+1}{3^{k+2}} \cdot \frac{3^{k+1}}{k}$$

$$= \lim_{k \to +\infty} \frac{k+1}{3k} = \frac{1}{3} < 1,$$

therefore the series converges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$ converges or diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k}$$

$$= \lim_{k \to +\infty} \frac{(2k+2)^{k+3}}{(k+2)!} \cdot \frac{(k+1)!}{(2k)^{k+2}}$$

$$= \lim_{k \to +\infty} \frac{(2k+2)^{k+2}(2k+2)}{(k+2)!} \cdot \frac{(k+1)!}{(2k)^{k+2}}$$

$$= \lim_{k \to +\infty} \frac{(2k+2)^{k+2}(2k+2)(k+1)!}{(2k)^{k+2}(k+2)!}$$

$$= \lim_{k \to +\infty} \left(1 + \frac{1}{k}\right)^{k+2} \frac{2k+2}{k+2} = 2e > 1,$$

therefore the series diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k}$ converges or diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{(2k)!}{3^k}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{(2k+2)!}{3^{k+1}} \cdot \frac{3^k}{(2k)!}$$

$$= \lim_{k\to+\infty} \frac{(2k+1)(2k+2)}{3} = +\infty,$$

therefore the series diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{3k+4}$ converges or diverges.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{3k+4}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k \to +\infty} \frac{u_{k+1}}{u_k} = \lim_{k \to +\infty} \frac{3k+4}{3k+7} = 1,$$

therefore the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the ratio test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{3k+4}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim_{k\to+\infty} \frac{3k+4}{3k+7} = 1,$$

therefore the series may converge or diverge, so that another test must be tried. For example, by the integral test,

$$\int_1^{+\infty} \frac{dx}{3x+4} = \lim_{\ell\to+\infty} \int_1^{\ell} \frac{dx}{3x+4}$$

$$= \lim_{\ell\to+\infty} \frac{1}{3}\ln(3x+4)\Big]_1^{\ell} = +\infty,$$

therefore the series diverges.

THEOREM (**The Root Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k}$$

(a) If $\rho < 1$, the series converges.

(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.

(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the root test to determine whether the series $\displaystyle\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^{k}$ converges or diverges.

EXAMPLE: Use the root test to determine whether the series $\sum_{k=2}^{\infty}\left(\frac{4k-5}{2k+1}\right)^k$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k} = \lim_{k\to+\infty} \frac{4k-5}{2k+1} = 2 > 1,$$

therefore the series diverges.

EXAMPLE: Use the root test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$ converges or diverges.

EXAMPLE: Use the root test to determine whether the series $\sum_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$ converges or diverges.

SOLUTION:

We have

$$\rho = \lim_{k \to +\infty} \sqrt[k]{u_k} = \lim_{k \to +\infty} \frac{1}{\ln(k+1)} = 0 < 1,$$

therefore the series converges.