10.6 The Comparison, Ratio, and Root Tests

THEOREM (**The Comparison Test**): Let $\sum\limits_{k=1}^{\infty} a_k$ and $\sum\limits_{k=1}^{\infty} b_k$ be series with nonnegative terms and suppose that $a_1 \le b_1,\ a_2 \le b_2,\ a_3 \le b_3, \ldots, a_k \le b_k, \ldots$

(a) If the "bigger series" $\sum\limits_{k=1}^{\infty} b_k$ converges, then the "smaller series" $\sum\limits_{k=1}^{\infty} a_k$ also converges.

(b) If the "smaller series" $\sum\limits_{k=1}^{\infty} a_k$ diverges, then the "bigger series" $\sum\limits_{k=1}^{\infty} b_k$ also diverges.

EXAMPLE: Use the comparison test to determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}-1}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$ (c) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1}$

SOLUTION:

(a) Since $\displaystyle\frac{1}{\sqrt[3]{k}-1} > \frac{1}{\sqrt[3]{k}}$ and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}}$ diverges, it follows that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{\sqrt[3]{k}-1}$ also diverges.

(b) Since $\displaystyle\frac{1}{k^2+k+1} < \frac{1}{k^2}$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges, it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k+1}$ also converges.

(c) Since $\displaystyle\frac{1}{k^2-1} < \frac{2}{k^2}$ and $\displaystyle\sum_{k=2}^{\infty} \frac{2}{k^2} = 2\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, it follows that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{k^2-1}$ also converges.

THEOREM (**The Limit Comparison Test**): Let $\sum a_k$ and $\sum b_k$ be series with positive terms and suppose that

$$\rho = \lim_{k\to+\infty} \frac{a_k}{b_k}.$$

If ρ is finite and $\rho > 0$, then the series both converge or both diverge.

EXAMPLE: Use the limit comparison test to determine whether the following series converge or diverge:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ (b) $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k^2-k-1}$ (c) $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2-k+5}{k^5+k^4+2k-2}$

SOLUTION:

(a) Put $a_n = \dfrac{1}{\sqrt{k}+1}$, $b_n = \dfrac{1}{\sqrt{k}}$. Then $\displaystyle\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{\sqrt{k}}{\sqrt{k}+1} = \lim_{k\to+\infty} \frac{1}{1+\frac{1}{\sqrt{k}}} = 1$. Since $\rho = 1$ and $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges, it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}+1}$ also diverges.

(b) Put $a_n = \dfrac{1}{2k^2-k-1}$, $b_n = \dfrac{1}{2k^2}$. Then $\displaystyle\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{2k^2}{2k^2-k-1} = \lim_{k\to+\infty} \frac{1}{1-\frac{1}{2k}-\frac{1}{2k^2}} = 1$. Since $\rho = 1$ and $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k^2} = \frac{1}{2}\sum_{k=2}^{\infty} \frac{1}{k^2}$ converges, it follows that $\displaystyle\sum_{k=2}^{\infty} \frac{1}{2k^2-k-1}$ also converges.

(c) Put $a_n = \dfrac{4k^2-k+5}{k^5+k^4+2k-2}$, $b_n = \dfrac{4}{k^3}$. Then $\displaystyle\rho = \lim_{k\to+\infty} \frac{a_k}{b_k} = \lim_{k\to+\infty} \frac{k^3(4k^2-k+5)}{4(k^5+k^4+2k-2)} = \lim_{k\to+\infty} \frac{4k^5-k^4+5k^3}{4k^5+4k^4+8k-8} = \lim_{k\to+\infty} \frac{1-\frac{1}{4k}+\frac{5}{4k^2}}{1+\frac{1}{k}+\frac{2}{k^4}-\frac{2}{k^5}} = 1$. Since $\rho = 1$ and $\displaystyle\sum_{k=1}^{\infty} \frac{4}{k^3} = 4\sum_{k=1}^{\infty} \frac{1}{k^3}$ converges, it follows that $\displaystyle\sum_{k=1}^{\infty} \frac{4k^2-k+5}{k^5+k^4+2k-2}$ also converges.

THEOREM (**The Ratio Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to+\infty} \frac{u_{k+1}}{u_k}.$$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the ratio test to determine whether the following series converge or diverge:

(a) $\sum\limits_{k=1}^{\infty} \frac{k+1}{k!}$ (b) $\sum\limits_{k=1}^{\infty} \frac{k}{3^{k+1}}$ (c) $\sum\limits_{k=1}^{\infty} \frac{(2k)^{k+2}}{(k+1)!}$ (d) $\sum\limits_{k=1}^{\infty} \frac{(2k)!}{3^k}$ (e) $\sum\limits_{k=1}^{\infty} \frac{1}{3k+4}$

SOLUTION:

(a) We have $\rho = \lim\limits_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \frac{k+2}{(k+1)!}\cdot\frac{k!}{k+1} = \lim\limits_{k\to+\infty} \frac{k+2}{(k+1)(k+1)} = 0 < 1$, therefore the series converges.

(b) We have $\rho = \lim\limits_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \frac{k+1}{3^{k+2}}\cdot\frac{3^{k+1}}{k} = \lim\limits_{k\to+\infty} \frac{k+1}{3k} = \frac{1}{3} < 1$, therefore the series converges.

(c) We have $\rho = \lim\limits_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \frac{(2k+2)^{k+3}}{(k+2)!}\cdot\frac{(k+1)!}{(2k)^{k+2}} = \lim\limits_{k\to+\infty} \frac{(2k+2)^{k+2}(2k+2)}{(k+2)!}\cdot\frac{(k+1)!}{(2k)^{k+2}} =$ $\lim\limits_{k\to+\infty} \frac{(2k+2)^{k+2}(2k+2)(k+1)!}{(2k)^{k+2}(k+2)!} = \lim\limits_{k\to+\infty} \left(1+\frac{1}{k}\right)^{k+2}\frac{2k+2}{k+2} = 2e > 1$, therefore the series diverges.

(d) We have $\rho = \lim\limits_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \frac{(2k+2)!}{3^{k+1}}\cdot\frac{3^k}{(2k)!} = \lim\limits_{k\to+\infty} \frac{(2k+1)(2k+2)}{3} = +\infty$, therefore the series diverges.

(e) We have $\rho = \lim\limits_{k\to+\infty} \frac{u_{k+1}}{u_k} = \lim\limits_{k\to+\infty} \frac{3k+4}{3k+7} = 1$, therefore the series may converge or diverge, so that another test must be tried. For example, by the integral test, $\int\limits_1^{+\infty} \frac{dx}{3x+4} = \lim\limits_{\ell\to+\infty} \int\limits_1^{\ell} \frac{dx}{3x+4} =$ $\lim\limits_{\ell\to+\infty} \frac{1}{3}\ln(3x+4)\Big]_1^{\ell} = +\infty$, therefore the series diverges.

THEOREM (**The Root Test**): Let $\sum u_k$ be a series with positive terms and suppose that

$$\rho = \lim_{k\to+\infty} \sqrt[k]{u_k}$$

(a) If $\rho < 1$, the series converges.
(b) If $\rho > 1$ or $\rho = +\infty$, the series diverges.
(c) If $\rho = 1$, the series may converge or diverge, so that another test must be tried.

EXAMPLE: Use the root test to determine whether the following series converge or diverge:

(a) $\sum\limits_{k=2}^{\infty} \left(\frac{4k-5}{2k+1}\right)^k$ (b) $\sum\limits_{k=1}^{\infty} \frac{1}{(\ln(k+1))^k}$

SOLUTION:

(a) We have $\rho = \lim\limits_{k\to+\infty} \sqrt[k]{u_k} = \lim\limits_{k\to+\infty} \frac{4k-5}{2k+1} = 2 > 1$, therefore the series diverges.

(b) We have $\rho = \lim\limits_{k\to+\infty} \sqrt[k]{u_k} = \lim\limits_{k\to+\infty} \frac{1}{\ln(k+1)} = 0 < 1$, therefore the series converges.