

DEFINITION: An **infinite series** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots + u_k + \dots$$

The numbers u_1, u_2, u_3, \dots are called the **terms** of the series.

Consider

$$s_1 = u_1$$

$$s_2 = u_1 + u_2$$

$$s_3 = u_1 + u_2 + u_3$$

...

$$s_n = u_1 + u_2 + u_3 + \dots + u_n = \sum_{k=1}^n u_k$$

DEFINITION: Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \dots + u_k + \dots$$

If the sequence $\{s_n\}$ converges to a limit S , then the series is said to **converge** to S , and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to **diverge**. A divergent series has no sum.

THEOREM: A geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^k + \dots \quad (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

EXAMPLE: Evaluate

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = 5 + \frac{5}{4} + \frac{5}{4^2} + \dots + \frac{5}{4^k} + \dots$$

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We have $a = 5$, $r = \frac{1}{4}$. Since $|r| = \frac{1}{4} < 1$, the series converges and the sum is

$$\sum_{k=0}^{\infty} \frac{5}{4^k} = \frac{a}{1-r} = \frac{5}{1-\frac{1}{4}} = \frac{20}{3}.$$

EXAMPLE: Evaluate $\sum_{k=1}^{\infty} 3^{2k} 5^{1-k}$

SOLUTION: First, compare this series with

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \dots + ar^k + \dots$$

We have

$$\sum_{k=1}^{\infty} 3^{2k} 5^{1-k} = \sum_{k=1}^{\infty} \frac{9^k}{5^{k-1}} = \sum_{k=1}^{\infty} 9 \left(\frac{9}{5}\right)^{k-1}$$

therefore $r = \frac{9}{5} > 1$, so the series diverges.

TELESCOPING SUMS:

Find

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots$$

We have

$$\begin{aligned} s_n &= \sum_{k=1}^n \frac{1}{k(k+1)} \\ &= \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \frac{1}{4 \cdot 5} + \dots + \frac{1}{n(n+1)} \\ &= \left[\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1} \right] \\ &= \left(1 - \frac{1}{2} \right) + \left(\frac{1}{2} - \frac{1}{3} \right) + \left(\frac{1}{3} - \frac{1}{4} \right) \\ &\quad + \left(\frac{1}{4} - \frac{1}{5} \right) + \dots + \left(\frac{1}{n} - \frac{1}{n+1} \right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2} \right) + \left(-\frac{1}{3} + \frac{1}{3} \right) + \left(-\frac{1}{4} + \frac{1}{4} \right) \\ &\quad + \dots + \left(-\frac{1}{n} + \frac{1}{n} \right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Therefore

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n \rightarrow +\infty} s_n = \lim_{n \rightarrow +\infty} \left(1 - \frac{1}{n+1} \right) = 1$$

THEOREM (The Divergence Test):

(a) If $\lim_{k \rightarrow +\infty} u_k \neq 0$, then the series $\sum u_k$ diverges.

(b) If $\lim_{k \rightarrow +\infty} u_k = 0$, then the series $\sum u_k$ may either converge or diverge.

THEOREM: If the series $\sum u_k$ converges, then $\lim_{k \rightarrow +\infty} u_k = 0$.

THEOREM:

(a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty} (u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$
$$\sum_{k=1}^{\infty} (u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) If c is a nonzero constant, then the series $\sum u_k$ and $\sum cu_k$ both converge or both diverge. In the case of convergence, the sums are related by

$$\sum_{k=1}^{\infty} cu_k = c \sum_{k=1}^{\infty} u_k$$

(c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K , the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \dots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \dots$$

both converge or both diverge.

THEOREM (The Integral Test): Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty)$, then

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x) dx$$

both converge or both diverge.

THEOREM (**Convergence of p -Series**):

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots + \frac{1}{k^p} + \dots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.