10.4 Infinite Series

DEFINITION: An **infinite series** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots + u_k + \ldots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the **terms** of the series.

Consider

$$\begin{aligned} s_1 &= u_1 \\ s_2 &= u_1 + u_2 \\ s_3 &= u_1 + u_2 + u_3 \\ &\ldots \\ s_n &= u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k \end{aligned}$$

DEFINITION: Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \ldots + u_k + \ldots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to **converge** to S, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to **diverge.** A divergent series has no sum.

THEOREM: A geometric series

$$\sum_{k=0}^{\infty} ar^k = a + ar + ar^2 + \ldots + ar^k + \ldots \quad (a \neq 0)$$

converges if $|r| < 1$ and diverges if $|r| \geq 1$. If the series converges, then the sum is

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1-r}$$

TELESCOPING SUMS: Find $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots$

We have

$$\begin{aligned} s_n = \sum_{k=1}^{n} \frac{1}{k(k+1)} &= \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \ldots + \frac{1}{n(n+1)} \\ &= \left[\frac{1}{k(k+1)} = \frac{1}{k} - \frac{1}{k+1}\right] \\ &= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \ldots + \left(\frac{1}{n} - \frac{1}{n+1}\right) \\ &= 1 + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \frac{1}{4}\right) + \ldots + \left(-\frac{1}{n} + \frac{1}{n}\right) - \frac{1}{n+1} \\ &= 1 - \frac{1}{n+1} \end{aligned}$$

Therefore $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k(k+1)} = \lim_{n\to+\infty} s_n = \lim_{n\to+\infty} \left(1 - \frac{1}{n+1}\right) = 1$

10.5 Convergence Tests

THEOREM (**The Divergence Test**):

(a) If $\lim_{k\to+\infty} u_k \neq 0$, then the series $\sum u_k$ diverges.

(b) If $\lim_{k\to+\infty} u_k = 0$, then the series $\sum u_k$ may either converge or diverge.

THEOREM: If the series $\sum u_k$ converges, then $\lim_{k\to+\infty} u_k = 0$.

THEOREM:

(a) If $\sum u_k$ and $\sum v_k$ are convergent series, then $\sum(u_k + v_k)$ and $\sum(u_k - v_k)$ are convergent series and the sums of these series are related by

$$\sum_{k=1}^{\infty}(u_k + v_k) = \sum_{k=1}^{\infty} u_k + \sum_{k=1}^{\infty} v_k$$

$$\sum_{k=1}^{\infty}(u_k - v_k) = \sum_{k=1}^{\infty} u_k - \sum_{k=1}^{\infty} v_k$$

(b) If c is a nonzero constant, then the series $\sum u_k$ and $\sum cu_k$ both converge or both diverge. In the case of convergence, the sums are related by

$$\sum_{k=1}^{\infty} cu_k = c\sum_{k=1}^{\infty} u_k$$

(c) Convergence or divergence is unaffected by deleting a finite number of terms from a series; in particular, for any positive integer K, the series

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots$$

$$\sum_{k=K}^{\infty} u_k = u_K + u_{K+1} + u_{K+2} + \ldots$$

both converge or both diverge.

THEOREM (**The Integral Test**): Let $\sum u_k$ be a series with positive terms, and let $f(x)$ be the function that results when k is replaced by x in the general term of the series. If f is decreasing and continuous on the interval $[a, +\infty)$, then

$$\sum_{k=1}^{\infty} u_k \quad \text{and} \quad \int_a^{+\infty} f(x)dx$$

both converge or both diverge.

THEOREM (**Convergence of p-Series**):

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \ldots + \frac{1}{k^p} + \ldots$$

converges if $p > 1$ and diverges if $0 < p \leq 1$.