10.3 Monotone Sequences

DEFINITION: A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing *if*	$a_1 < a_2 < a_3 < \ldots < a_n < \ldots$
increasing *if*	$a_1 \leq a_2 \leq a_3 \leq \ldots \leq a_n \leq \ldots$
strictly decreasing *if*	$a_1 > a_2 > a_3 > \ldots > a_n > \ldots$
decreasing *if*	$a_1 \geq a_2 \geq a_3 \geq \ldots \geq a_n \geq \ldots$

Moreover, a sequence that is either strictly increasing or strictly decreasing is called **strictly monotone**, and a sequence that is either increasing or decreasing is called **monotone.**

THEOREM: If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:

(a) There is a constant M, called an **upper bound** for the sequence, such that $a_n < M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.

(b) No upper bound exists, in which case $\lim\limits_{n\to+\infty} a_n = +\infty$.

THEOREM: If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:

(a) There is a constant M, called an **lower bound** for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.

(b) No lower bound exists, in which case $\lim\limits_{n\to+\infty} a_n = -\infty$.

Three Tests

DIFFERENCE BETWEEN SUCCESSIVE TERMS	CLASSIFICATION
$a_{n+1} - a_n > 0$	Strictly increasing
$a_{n+1} - a_n < 0$	Strictly decreasing
$a_{n+1} - a_n \geq 0$	Increasing
$a_{n+1} - a_n \leq 0$	Decreasing

RATIO OF SUCCESSIVE TERMS	CONCLUSION
$a_{n+1}/a_n > 1$	Strictly increasing
$a_{n+1}/a_n < 1$	Strictly decreasing
$a_{n+1}/a_n \geq 1$	Increasing
$a_{n+1}/a_n \leq 1$	Decreasing

DERIVATIVE OF f FOR $x \geq 1$	CONCLUSION FOR THE SEQUENCE WITH $a_n = f(n)$
$f'(x) > 0$	Strictly increasing
$f'(x) < 0$	Strictly decreasing
$f'(x) \geq 0$	Increasing
$f'(x) \leq 0$	Decreasing

10.4 Infinite Series

DEFINITION: An **infinite series** is an expression that can be written in the form

$$\sum_{k=1}^{\infty} u_k = u_1 + u_2 + u_3 + \ldots + u_k + \ldots$$

The numbers $u_1, u_2, u_3, \ldots$ are called the **terms** of the series.

Consider

$$\begin{aligned}
&s_1 = u_1 \\
&s_2 = u_1 + u_2 \\
&s_3 = u_1 + u_2 + u_3 \\
&\ldots \\
&s_n = u_1 + u_2 + u_3 + \ldots + u_n = \sum_{k=1}^{n} u_k
\end{aligned}$$

DEFINITION: Let $\{s_n\}$ be the sequence of partial sums of the series

$$u_1 + u_2 + u_3 + \ldots + u_k + \ldots$$

If the sequence $\{s_n\}$ converges to a limit S, then the series is said to **converge** to S, and S is called the **sum** of the series. We denote this by writing

$$S = \sum_{k=1}^{\infty} u_k$$

If the sequence of partial sums diverges, then the series is said to **diverge.** A divergent series has no sum.