Stated formally, an **infinite sequence,** or more simply a **sequence,** is an unending succession of numbers, called **terms**.

EXAMPLE:

(a) $0,\ 0,\ 0,\ 0,\ldots$

(d) $0,\ 1,\ 0,\ 1,\ 0,\ 1,\ 0,\ 1,\ldots$

(b) $1,\ 2,\ 3,\ 4,\ 5,\ldots$

(e) $1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ldots$

(c) $2,\ 4,\ 6,\ 8,\ 10,\ldots$

(f) $1,\ 10,\ -5,\ 3,\ 99,\ 100,\ -23,\ldots$

DEFINITION: A **sequence** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n$, $n = 1, 2, 3, \ldots$

EXAMPLE:

$$a_n = n \quad \text{or} \quad \{n\}_{n=1}^{+\infty} \quad \text{or} \quad \{n\}$$

which means

$$1,\ 2,\ 3,\ 4,\ 5, \ldots$$

EXAMPLE:

$$a_n = \frac{1}{2^{n-1}} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}_{n=1}^{+\infty} \quad \text{or} \quad \left\{\frac{1}{2^{n-1}}\right\}$$

which means

$$1,\ \frac{1}{2},\ \frac{1}{4},\ \frac{1}{8},\ \frac{1}{16}, \ldots$$

EXAMPLE:

$$a_n = (-1)^{n+1}\frac{n}{2n+1} \quad \text{or} \quad \left\{(-1)^{n+1}\frac{n}{2n+1}\right\}_{n=1}^{+\infty}$$

or

$$\left\{(-1)^{n+1}\frac{n}{2n+1}\right\}$$

which means

$$\frac{1}{3},\ -\frac{2}{5},\ \frac{3}{7},\ -\frac{4}{9},\ldots$$

DEFINITION: A sequence $\{a_n\}$ is said to **converge** to the **limit** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \to +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to **diverge**.

THEOREM: Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2 respectively. Then

$$(a) \quad \lim_{n\to+\infty} c = c$$

$$(b) \quad \lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n = cL_1$$

$$(c) \quad \lim_{n\to+\infty} (a_n \pm b_n) = \lim_{n\to+\infty} a_n \pm \lim_{n\to+\infty} b_n = L_1 \pm L_2$$

$$(e) \quad \lim_{n\to+\infty} (a_n b_n) = \lim_{n\to+\infty} a_n \cdot \lim_{n\to+\infty} b_n = L_1 L_2$$

$$(f) \quad \lim_{n\to+\infty} \left(\frac{a_n}{b_n}\right) = \frac{\lim\limits_{n\to+\infty} a_n}{\lim\limits_{n\to+\infty} b_n} = \frac{L_1}{L_2} \quad (if\ L_2 \neq 0)$$

THEOREM: A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L.

THEOREM (The Squeezing Theorem for Sequences):
Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that

$$a_n \le b_n \le c_n$$

for all values of n beyond some index N. If

$$\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = L$$

then

$$\lim_{n\to+\infty} b_n = L$$

In other words, if the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \to +\infty$, then $\{b_n\}$ also has the limit L as $n \to +\infty$.

DEFINITION: A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $a_1 < a_2 < \ldots < a_n < \ldots$

increasing if $a_1 \leq a_2 \leq \ldots \leq a_n \leq \ldots$

strictly decreasing if $a_1 > a_2 > \ldots > a_n > \ldots$

decreasing if $a_1 \geq a_2 \geq \ldots \geq a_n \geq \ldots$

Moreover, a sequence that is either strictly increasing or strictly decreasing is called **strictly monotone**, and a sequence that is either increasing or decreasing is called **monotone.**

THEOREM: If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:

(a) There is a constant M, called an **upper bound** for the sequence, such that $a_n < M$ for all n, in which case the sequence converges to a limit L satisfying $L \leq M$.

(b) No upper bound exists, in which case

$$\lim_{n \to +\infty} a_n = +\infty.$$

THEOREM: If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:

(a) There is a constant M, called an **lower bound** for the sequence, such that $a_n \geq M$ for all n, in which case the sequence converges to a limit L satisfying $L \geq M$.

(b) No lower bound exists, in which case

$$\lim_{n \to +\infty} a_n = -\infty.$$