

10.2 Sequences

Stated formally, an **infinite sequence**, or more simply a **sequence**, is an unending succession of numbers, called **terms**.

EXAMPLE:

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|-----------------------------------------------------------------------------------------|------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <p>(a) 0, 0, 0, 0, ...</p> <p>(b) 1, 2, 3, 4, 5, ...</p> <p>(c) 2, 4, 6, 8, 10, ...</p> | <p>(d) 0, 1, 0, 1, 0, 1, 0, 1, ...</p> <p>(e) $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \dots$</p> <p>(f) 1, 10, -5, 3, 99, 100, -23, ...</p> |
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DEFINITION: A **sequence** is a function whose domain is a set of integers. Specifically, we will regard the expression $\{a_n\}_{n=1}^{+\infty}$ to be an alternative notation for the function $f(n) = a_n, n = 1, 2, 3, \dots$

EXAMPLE:

$$a_n = n \quad \text{or} \quad \{n\}_{n=1}^{+\infty} \quad \text{or} \quad \{n\}$$

which means 1, 2, 3, 4, 5, ...

EXAMPLE:

$$a_n = \frac{1}{2^{n-1}} \quad \text{or} \quad \left\{ \frac{1}{2^{n-1}} \right\}_{n=1}^{+\infty} \quad \text{or} \quad \left\{ \frac{1}{2^{n-1}} \right\}$$

which means $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots$

EXAMPLE:

$$a_n = (-1)^{n+1} \frac{n}{2n+1} \quad \text{or} \quad \left\{ (-1)^{n+1} \frac{n}{2n+1} \right\}_{n=1}^{+\infty} \quad \text{or} \quad \left\{ (-1)^{n+1} \frac{n}{2n+1} \right\}$$

which means $\frac{1}{3}, -\frac{2}{5}, \frac{3}{7}, -\frac{4}{9}, \dots$

DEFINITION: A sequence $\{a_n\}$ is said to **converge** to the **limit** L if given any $\epsilon > 0$, there is a positive integer N such that $|a_n - L| < \epsilon$ for $n \geq N$. In this case we write

$$\lim_{n \rightarrow +\infty} a_n = L$$

A sequence that does not converge to some finite limit is said to **diverge**.

THEOREM: Suppose that the sequences $\{a_n\}$ and $\{b_n\}$ converge to limits L_1 and L_2 respectively. Then

- (a) $\lim_{n \rightarrow +\infty} c = c$
- (b) $\lim_{n \rightarrow +\infty} ca_n = c \lim_{n \rightarrow +\infty} a_n = cL_1$
- (c) $\lim_{n \rightarrow +\infty} (a_n \pm b_n) = \lim_{n \rightarrow +\infty} a_n \pm \lim_{n \rightarrow +\infty} b_n = L_1 \pm L_2$
- (e) $\lim_{n \rightarrow +\infty} (a_n b_n) = \lim_{n \rightarrow +\infty} a_n \cdot \lim_{n \rightarrow +\infty} b_n = L_1 L_2$
- (f) $\lim_{n \rightarrow +\infty} \left(\frac{a_n}{b_n} \right) = \frac{\lim_{n \rightarrow +\infty} a_n}{\lim_{n \rightarrow +\infty} b_n} = \frac{L_1}{L_2} \quad (\text{if } L_2 \neq 0)$

THEOREM: A sequence converges to a limit L if and only if the sequences of even-numbered terms and odd-numbered terms both converge to L .

THEOREM (The Squeezing Theorem for Sequences): Let $\{a_n\}$, $\{b_n\}$, and $\{c_n\}$ be sequences such that

$$a_n \leq b_n \leq c_n$$

for all values of n beyond some index N . If

$$\lim_{n \rightarrow +\infty} a_n = \lim_{n \rightarrow +\infty} c_n = L$$

then

$$\lim_{n \rightarrow +\infty} b_n = L$$

In other words, if the sequences $\{a_n\}$ and $\{c_n\}$ have a common limit L as $n \rightarrow +\infty$, then $\{b_n\}$ also has the limit L as $n \rightarrow +\infty$.

10.3 Monotone Sequences

DEFINITION: A sequence $\{a_n\}_{n=1}^{+\infty}$ is called

strictly increasing if $a_1 < a_2 < a_3 < \dots < a_n < \dots$

increasing if $a_1 \leq a_2 \leq a_3 \leq \dots \leq a_n \leq \dots$

strictly decreasing if $a_1 > a_2 > a_3 > \dots > a_n > \dots$

decreasing if $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq \dots$

Moreover, a sequence that is either strictly increasing or strictly decreasing is called **strictly monotone**, and a sequence that is either increasing or decreasing is called **monotone**.

THEOREM: If a sequence $\{a_n\}$ is eventually increasing, then there are two possibilities:

(a) There is a constant M , called an **upper bound** for the sequence, such that $a_n < M$ for all n , in which case the sequence converges to a limit L satisfying $L \leq M$.

(b) No upper bound exists, in which case $\lim_{n \rightarrow +\infty} a_n = +\infty$.

THEOREM: If a sequence $\{a_n\}$ is eventually decreasing, then there are two possibilities:

(a) There is a constant M , called an **lower bound** for the sequence, such that $a_n \geq M$ for all n , in which case the sequence converges to a limit L satisfying $L \geq M$.

(b) No lower bound exists, in which case $\lim_{n \rightarrow +\infty} a_n = -\infty$.