DEFINITION: If f can be differentiated n times at x_0, then we define the **n-th Taylor polynomial for f about $x = x_0$** to be

$$p_n(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$$

In the special case where $x_0 = 0$, this polynomial becomes

$$p_n(x) = f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\dots+\frac{f^{(n)}(0)}{n!}x^n$$

in which case we call it the **n-th Maclaurin polynomial for f**

DEFINITION: If f has derivatives of all orders at x_0, then we define the **Taylor series for f about $x = x_0$** to be

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k$$

$$= f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2$$

$$+\frac{f'''(x_0)}{3!}(x-x_0)^3 + \ldots + \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k + \ldots$$

In the special case where $x_0 = 0$, this series becomes

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!}x^k$$

$$= f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots$$

$$+\frac{f^{(k)}(0)}{k!}x^k + \ldots$$

in which case we call it the **Maclaurin series for f**.

Put

$$R_n(x) = f(x) - p_n(x) = f(x) - \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

THEOREM: The equality

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k$$

holds at a number x if and only if

$$\lim_{n \to +\infty} R_n(x) = 0.$$

THEOREM (**The Remainder Estimation Theorem**): If the function f can be differentiated $n+1$ times on an interval I containing the number x_0, and if M is an upper bound for $|f^{(n+1)}(x)|$ on I, that is, $|f^{(n+1)}(x)| \leq M$ for all x in I, then

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x - x_0|^{n+1}$$

for all x in I.

MCLAURIN SERIES

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + x^3 + \dots$$

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} = 1 - x^2 + x^4 - x^6 + \dots$$

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

$$\sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$\cos x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

$$\ln(1+x) = \sum_{k=1}^{\infty} (-1)^{k+1} \frac{x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

$$\tan^{-1} x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

$$(1+x)^m = 1 + \sum_{k=1}^{\infty} \frac{m(m-1)\dots(m-k+1)}{k!} x^k$$