10.10 Differentiation and Integration of Power Series

THEOREM (**Differentiation of Power Series**): Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad (x_0 - R < x < x_0 + R)$$

Then:

(a) The function f is differentiable on the interval $(x_0 - R, x_0 + R)$.

(b) If the power series representation for f is differentiated term by term, then the resulting series has radius of convergence R and converges to f' on the interval $(x_0 - R, x_0 + R)$; that is

$$f'(x) = \sum_{k=0}^{\infty} \frac{d}{dx}[c_k (x - x_0)^k] \quad (x_0 - R < x < x_0 + R)$$

THEOREM (**Integration of Power Series**): Suppose that a function f is represented by a power series in $x - x_0$ that has a nonzero radius of convergence R; that is,

$$f(x) = \sum_{k=0}^{\infty} c_k (x - x_0)^k \quad (x_0 - R < x < x_0 + R)$$

(a) If the power series representation of f is integrated term by term, then the resulting series has radius of convergence R and converges to an antiderivative for $f(x)$ on the interval $(x_0 - R, x_0 + R)$; that is

$$\int f(x)dx = \sum_{k=0}^{\infty} \left[\frac{c_k}{k+1}(x - x_0)^{k+1}\right] + C \quad (x_0 - R < x < x_0 + R)$$

(b) If α and β are points in the interval $(x_0 - R, x_0 + R)$, and if the power series representation of f is integrated term by term from α to β, then the resulting series converges absolutely on the interval $(x_0 - R, x_0 + R)$ and

$$\int_{\alpha}^{\beta} f(x)dx = \sum_{k=0}^{\infty} \left[\int_{\alpha}^{\beta} c_k (x - x_0)^k dx\right]$$