Section 6.5 - The Definite Integral

DEFINITION: A function f is said to be **integrable** on a finite interval $[a, b]$ if the limit

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

exists and does not depend on the choice of the partitions or on the choice of the numbers x_k^* in the subintervals. When this is the case we denote the limit by the symbol

$$\int_a^b f(x)dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k$$

which is called the **definite integral** of f from a to b. The numbers a and b are called the **lower limit of integration** and the **upper limit of integration**, respectively, and $f(x)$ is called the **integrand**.

EXAMPLE: Let $f(x) = 1$, then f is integrable on $[a, b]$ and

$$\int_a^b f(x)dx = b - a.$$

PROOF: We have

$$\sum_{k=1}^{n} f(x_k^*) \Delta x_k = \sum_{k=1}^{n} 1 \cdot \Delta x_k = \sum_{k=1}^{n} \Delta x_k = b - a,$$

therefore

$$\lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*) \Delta x_k = \lim_{\max \Delta x_k \to 0} (b - a) = b - a.$$

THEOREM: If a function f is continuous on an interval $[a, b]$, then f is integrable on $[a, b]$.

THEOREM (**The Fundamental Theorem Of Calculus**): If f is continuous on $[a, b]$ and F is any antiderivative of f on $[a, b]$, then

$$\int_a^b f(x)dx = F(b) - F(a)$$

DEFINITION: A function F is called an **antiderivative** of a function f on a given interval I if $F'(x) = f(x)$ for all x in the interval.

EXAMPLE: If $f(x) = 1$, then $x,\ x + 1, x - 2, x + 100, \ldots$ are antiderivatives of f.

THEOREM: If $F(x)$ is any antideriavtive of $f(x)$ on an interval I, then for any constant C the function $F(x)+C$ is also an antiderivative on that interval. Moreover, each antiderivative of $f(x)$ on the interval I can be expressed in the form $F(x) + C$ choosing the constant C appropriately.

NOTATION: Denote

$$\int f(x)dx = F(x) + C$$

which is called the indefinite integral.

EXAMPLE:

$f(x)$	$F(x)$	$\int f(x)dx = F(x) + C$	$\int_5^9 f(x)dx = F(b) - F(a)$
1	$x, \quad x+1, \quad x-100, \ldots$	$x + C$	$9 - 5 = 4$
2	$2x, \quad 2x-5, \quad 2x+87, \ldots$	$2x + C$	$2 \cdot 9 - 2 \cdot 5 = 8$
10	$10x, \quad 10x-2, \quad 10x+10, \ldots$	$10x + C$	$10 \cdot 9 - 10 \cdot 5 = 40$
x	$\frac{x^2}{2}, \quad \frac{x^2}{2} - 3, \quad \frac{x^2}{2} + 23, \ldots$	$\frac{x^2}{2} + C$	$\frac{9^2}{2} - \frac{5^2}{2} = 28$
$x+1$	$\frac{x^2}{2} + x, \quad \frac{x^2}{2} + x - 12, \quad \frac{x^2}{2} + x - 47, \ldots$	$\frac{x^2}{2} + x + C$	$\frac{9^2}{2} + 9 - \left(\frac{5^2}{2} + 5\right) = 32$
x^3	$\frac{x^3}{3}, \quad \frac{x^2}{2} - 4, \quad \frac{x^2}{2} + 9, \ldots$	$\frac{x^3}{3} + C$	$\frac{9^3}{3} - \frac{5^3}{3} = 202$
$\sqrt{x}$	$\frac{2x^{3/2}}{3}, \quad \frac{2x^{3/2}}{3} - 5, \quad \frac{2x^{3/2}}{3} + 3, \ldots$	$\frac{2x^{3/2}}{3} + C$	$\frac{2 \cdot 9^{3/2}}{3} - \frac{2 \cdot 5^{3/2}}{3} \approx 10.546$

TABLE

DIFFERENTIATION FORMULA	INTEGRATION FORMULA
$[x^r]' = (r-1)x^r \quad (r \neq -1)$	$\int x^r dx = \frac{x^{r+1}}{r+1} + C \quad (r \neq -1)$
$[\sin x]' = \cos x$	$\int \sin x dx = -\cos x + C$
$[\cos x]' = -\sin x + C$	$\int \cos x dx = \sin x + C$
$[e^x]' = e^x$	$\int e^x dx = e^x + C$
$[a^x]' = a^x \ln a \quad (a > 0,\ a \neq 1)$	$\int a^x dx = \frac{a^x}{\ln a} + C \quad (a > 0,\ a \neq 1)$
$[\ln \lvert x \rvert]' = \frac{1}{x}$	$\int \frac{1}{x} dx = \ln \lvert x \rvert + C$

DEFINITION:

(a) If a is in the domain of f, we define

$$\int_a^a f(x)dx = 0$$

(b) If f is integrable on $[a, b]$, then we define

$$\int_a^b f(x)dx = -\int_b^a f(x)dx$$

THEOREM: If f and g are integrable on $[a, b]$ and if c is a constant, then cf, $f + g$, and $f - g$ are integrable on $[a, b]$ and

(a) $\displaystyle\int\limits_a^b cf(x)dx = c\int\limits_a^b f(x)dx$

(b) $\displaystyle\int\limits_a^b [f(x) + g(x)]dx = \int\limits_a^b f(x)dx + \int\limits_a^b g(x)dx$

(c) $\displaystyle\int\limits_a^b [f(x) - g(x)]dx = \int\limits_a^b f(x)dx - \int\limits_a^b g(x)dx$

THEOREM: If f is integrable on a closed interval containing the three numbers a, b, and c, then

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

no matter how the numbers are ordered.

THEOREM:

(a) If f is integrable on $[a, b]$ and $f(x) \geq 0$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq 0$$

(b) If f and g are integrable on $[a, b]$ and $f(x) \geq g(x)$ for all x in $[a, b]$, then

$$\int_a^b f(x)dx \geq \int_a^b g(x)dx$$