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and the total area A_n of the region R_n will be

$$A_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

REMARK: We have

$$
\begin{aligned}
x_k^* &= x_{k-1} = a + (k-1)\Delta x \quad \text{if } x_k^* \text{ is the left endpoint} \\
x_k^* &= x_k = a + k\Delta x \quad \text{if } x_k^* \text{ is the right endpoint} \\
x_k^* &= \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x \quad \text{if } x_k^* \text{ is the midpoint}
\end{aligned}
\qquad (*)
$$

DEFINITION (**Area Under a Curve**): If a function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*) \Delta x$$

EXAMPLE: Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$ with $n = 10,\ n = 20$, and $n = 50$.

EXAMPLE: Use the Definition with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = 9 - x^2$ and over the interval $[0, 3]$.

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SOLUTION: Each subinterval will have length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

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and from $(*)$

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

EXAMPLE: Use the Definition with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = 9 - x^2$ and over the interval $[0, 3]$.

SOLUTION: Each subinterval will have length

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and from (∗)

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned} f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2 \left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n} \left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \right) \\
&= \sum_{k=1}^{n} \frac{27}{n} - \sum_{k=1}^{n} \frac{27}{n^3}k^2 + \sum_{k=1}^{n} \frac{27}{n^3}k - \sum_{k=1}^{n} \frac{27}{4n^3} \\
&= \frac{27}{n} \sum_{k=1}^{n} 1 - \frac{27}{n^3} \sum_{k=1}^{n} k^2 + \frac{27}{n^3} \sum_{k=1}^{n} k - \frac{27}{4n^3} \sum_{k=1}^{n} 1 \\
&= 27 \left[\frac{1}{n} \sum_{k=1}^{n} 1 - \frac{1}{n^3} \sum_{k=1}^{n} k^2 + \frac{1}{n^3} \sum_{k=1}^{n} k - \frac{1}{4n^3} \sum_{k=1}^{n} 1 \right] \\
&= 27 \left[\frac{1}{n}n - \frac{1}{n^3} \frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3} \frac{n(n+1)}{2} - \frac{1}{4n^3}n \right] \\
&= 27 \left[1 - \frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{2n^2} - \frac{1}{4n^2} \right]
\end{aligned}
$$

It follows that

$$
\begin{aligned}
A &= \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x \\
&= \lim_{n\to+\infty} 27\left[1 - \frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{2n^2} - \frac{1}{4n^2}\right] \\
&= 27\left[1 - \frac{1}{3} + 0 - 0\right] = 27\left[1 - \frac{1}{3}\right] = 27 \cdot \frac{2}{3} = 18
\end{aligned}
$$

EXAMPLE: Use the Definition with x_k^* as the right point of each subinterval to find the area under the parabola $y = x^2$ and over the interval $[0, 1]$.