Section 6.4 - Sigma Notation; Area as a Limit

THEOREM:

(a) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

Let $f(x)$ be a continuous function on $[a, b]$. We divide the interval $[a, b]$ into n subintervals by inserting $n - 1$ equally spaced points between a and b, say

$$x_1, \ x_2, \ \ldots, \ x_{n-1}$$

Each of these subintervals has width

$$\Delta x = \frac{b-a}{n}$$

In each subinterval we choose an x-value at which we evaluate the function f to determine the height of a rectangle over the interval. We denote these values by

$$x_1^*, \ x_2^*, \ \ldots, \ x_n^*$$

The areas of rectangles constructed over these intervals will be

$$f(x_1^*)\Delta x, \ f(x_2^*)\Delta x, \ \ldots, \ f(x_n^*)\Delta x$$

and the total area A_n of the region R_n will be

$$A_n = f(x_1^*)\Delta x + f(x_2^*)\Delta x + \ldots + f(x_n^*)\Delta x = \sum_{k=1}^{n} f(x_k^*)\Delta x$$

REMARK: We have

$$\begin{aligned} x_k^* &= x_{k-1} = a + (k-1)\Delta x \quad \text{if } x_k^* \text{ is the left endpoint} \\ x_k^* &= x_k = a + k\Delta x \quad \text{if } x_k^* \text{ is the right endpoint} \\ x_k^* &= \frac{1}{2}(x_{k-1} + x_k) = a + \left(k - \frac{1}{2}\right)\Delta x \quad \text{if } x_k^* \text{ is the midpoint} \end{aligned} \qquad (*)$$

DEFINITION (**Area Under a Curve**): If a function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n \to +\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$

EXAMPLE: Find the left endpoint, right endpoint, and midpoint approximations of the area under the curve $y = 9 - x^2$ over the interval $[0, 3]$ with $n = 10,\ n = 20$, and $n = 50$.

EXAMPLE: Use the Definition with x_k^* as the midpoint of each subinterval to find the area under the parabola $y = 9 - x^2$ and over the interval $[0, 3]$.

SOLUTION: Each subinterval will have length

$$\Delta x = \frac{b-a}{n} = \frac{3-0}{n} = \frac{3}{n}$$

and from $(*)$

$$x_k^* = a + \left(k - \frac{1}{2}\right)\Delta x = \left(k - \frac{1}{2}\right)\left(\frac{3}{n}\right)$$

Thus,

$$\begin{aligned} f(x_k^*)\Delta x = [9 - (x_k^*)^2]\Delta x &= \left[9 - \left(k - \frac{1}{2}\right)^2\left(\frac{3}{n}\right)^2\right]\left(\frac{3}{n}\right) \\ &= \left[9 - \left(k^2 - k + \frac{1}{4}\right)\left(\frac{9}{n^2}\right)\right]\left(\frac{3}{n}\right) \\ &= \frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3} \end{aligned}$$

and therefore

$$\begin{aligned} \sum_{k=1}^{n} f(x_k^*)\Delta x &= \sum_{k=1}^{n}\left(\frac{27}{n} - \frac{27}{n^3}k^2 + \frac{27}{n^3}k - \frac{27}{4n^3}\right) \\ &= \sum_{k=1}^{n}\frac{27}{n} - \sum_{k=1}^{n}\frac{27}{n^3}k^2 + \sum_{k=1}^{n}\frac{27}{n^3}k - \sum_{k=1}^{n}\frac{27}{4n^3} \\ &= \frac{27}{n}\sum_{k=1}^{n}1 - \frac{27}{n^3}\sum_{k=1}^{n}k^2 + \frac{27}{n^3}\sum_{k=1}^{n}k - \frac{27}{4n^3}\sum_{k=1}^{n}1 \\ &= 27\left[\frac{1}{n}\sum_{k=1}^{n}1 - \frac{1}{n^3}\sum_{k=1}^{n}k^2 + \frac{1}{n^3}\sum_{k=1}^{n}k - \frac{1}{4n^3}\sum_{k=1}^{n}1\right] \\ &= 27\left[\frac{1}{n}n - \frac{1}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{1}{n^3}\frac{n(n+1)}{2} - \frac{1}{4n^3}n\right] \\ &= 27\left[1 - \frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{2n^2} - \frac{1}{4n^2}\right] \end{aligned}$$

It follows that

$$\begin{aligned} A &= \lim_{n\to+\infty}\sum_{k=1}^{n} f(x_k^*)\Delta x \\ &= \lim_{n\to+\infty} 27\left[1 - \frac{(n+1)(2n+1)}{6n^2} + \frac{n+1}{2n^2} - \frac{1}{4n^2}\right] \\ &= 27\left[1 - \frac{1}{3} + 0 - 0\right] = 27\left[1 - \frac{1}{3}\right] = 27\cdot\frac{2}{3} = 18 \end{aligned}$$