Section 6.4 - Sigma Notation; Area as a Limit

NOTATION: If $f(k)$ is a function of k, and if m and n are integers such that $m \leq n$, then

$$\sum_{k=m}^{n} f(k)$$

denotes the following sum:

$$f(m) + \ldots + f(n)$$

So,

$$\sum_{k=m}^{n} f(k) = f(m) + \ldots + f(n)$$

EXAMPLE:

$$\sum_{k=1}^{5} k = 1 + 2 + 3 + 4 + 5$$

$$\sum_{k=3}^{6} (k+1)^3 = 4^3 + 5^3 + 6^3 + 7^3$$

$$\sum_{k=1}^{4} (2k+1) = 3 + 5 + 7 + 9$$

$$\sum_{k=-2}^{1} (3k-1) = -7 + (-4) + (-1) + 2$$

$$\sum_{k=1}^{4} (-1)^k k = -1 + 2 - 3 + 4$$

$$\sum_{k=2}^{4} k \cos\left(\frac{k\pi}{7}\right) = 2\cos\left(\frac{2\pi}{7}\right) + 3\cos\left(\frac{3\pi}{7}\right) + 4\cos\left(\frac{4\pi}{7}\right)$$

$$\sum_{k=1}^{5} 1 = 1 + 1 + 1 + 1 + 1$$

$$\sum_{k=4}^{4} k^2 = 4^2$$

REMARK: In is not essential to use k as the index of summation. For example,

$$\sum_{k=1}^{5} \frac{1}{k} = \sum_{n=1}^{5} \frac{1}{n} = \sum_{m=1}^{5} \frac{1}{m} = \sum_{j=1}^{5} \frac{1}{j}$$

REMARK: A sum can be written in more than one way. For example,

$$\sum_{n=3}^{8} n = \sum_{n=4}^{9} (n-1) = \sum_{n=5}^{10} (n-2)$$

$$\sum_{n=3}^{8} \frac{n}{n+1} = \sum_{n=2}^{7} \frac{n+1}{n+2} = \sum_{n=1}^{6} \frac{n+2}{n+3}$$

THEOREM:

(a) $\displaystyle\sum_{k=1}^{n} ca_k = c\sum_{k=1}^{n} a_k$

(b) $\displaystyle\sum_{k=1}^{n}(a_k + b_k) = \sum_{k=1}^{n} a_k + \sum_{k=1}^{n} b_k$

(c) $\displaystyle\sum_{k=1}^{n}(a_k - b_k) = \sum_{k=1}^{n} a_k - \sum_{k=1}^{n} b_k$

THEOREM:

(a) $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \ldots + n = \frac{n(n+1)}{2}$

(b) $\displaystyle\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(c) $\displaystyle\sum_{k=1}^{n} k^3 = 1^3 + 2^3 + \ldots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$

EXAMPLE: Find $\displaystyle\sum_{k=1}^{30} k(k+1)$.

SOLUTION: We have

$$\sum_{k=1}^{30} k(k+1) = \sum_{k=1}^{30}(k^2 + k) = \sum_{k=1}^{30} k^2 + \sum_{k=1}^{30} k = \frac{30 \cdot 31 \cdot 61}{6} + \frac{30 \cdot 31}{2} = 9920$$

THEOREM (**Area Under a Curve**): If a function f is continuous on $[a, b]$ and if $f(x) \geq 0$ for all x in $[a, b]$, then the area under the curve $y = f(x)$ over the interval $[a, b]$ is defined by

$$A = \lim_{n\to+\infty} \sum_{k=1}^{n} f(x_k^*)\Delta x$$