Section 5.7 - Newton's Method

EXAMPLE: We know that $\sqrt{2} = 1.41421356237...$ Let us do the following:

$$
\begin{array}{ll}
x_1 = 1 & x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 1.5 \\
x_2 = 1.5 & x_3 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.41666666667 \\
x_3 = 1.41666666667 & x_4 = \frac{1}{2}\left(1.41666666667 + \frac{2}{1.41666666667}\right) \approx 1.41421568627 \\
x_4 = 1.41421568627 & x_5 = \frac{1}{2}\left(1.41421568627 + \frac{2}{1.41421568627}\right) \approx 1.41421356237
\end{array}
$$

NEWTON'S METHOD: Suppose that we are trying to find a root r of the equation $f(x) = 0$ and suppose that by some method we are able to obtain an initial rough estimate x_1 of r. Then

$$
r = \lim_{n\to\infty} x_n, \quad \text{where} \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 1, 2, 3, \ldots
$$

EXAMPLE: Use Newton's Method to approximate $\sqrt{2}$.

SOLUTION: Since $\sqrt{2}$ is the solution of the equation $x^2 - 2 = 0$, we have

$$
f(x) = x^2 - 2 \Longrightarrow f'(x) = 2x
$$

so

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 2}{2x_n} = x_n - \frac{x_n^2}{2x_n} + \frac{2}{2x_n} = x_n - \frac{x_n}{2} + \frac{1}{x_n} = \frac{x_n}{2} + \frac{1}{x_n} = \frac{1}{2}\left(x_n + \frac{2}{x_n}\right)
$$

Put $x_1 = 1$ the initial rough estimate of $\sqrt{2}$. Then

$$
\begin{array}{ll}
x_1 = 1 & x_2 = \frac{1}{2}\left(1 + \frac{2}{1}\right) = 1.5 \\
x_2 = 1.5 & x_3 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right) \approx 1.41666666667 \\
x_3 = 1.41666666667 & x_4 = \frac{1}{2}\left(1.41666666667 + \frac{2}{1.41666666667}\right) \approx 1.41421568627 \\
x_4 = 1.41421568627 & x_5 = \frac{1}{2}\left(1.41421568627 + \frac{2}{1.41421568627}\right) \approx 1.41421356237 \\
x_4 = 1.41421356237 & x_5 = \frac{1}{2}\left(1.41421356237 + \frac{2}{1.41421356237}\right) \approx 1.41421356237
\end{array}
$$

EXAMPLE: Use Newton's Method to approximate $\sqrt[3]{5}$.

SOLUTION: Since $\sqrt[3]{5}$ is the solution of the equation $x^3 - 5 = 0$, we have

$$
f(x) = x^3 - 5 \Longrightarrow f'(x) = 3x^2
$$

so

$$
x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - 5}{3x_n^2} = x_n - \frac{x_n^3}{3x_n^2} + \frac{5}{3x_n^2} = x_n - \frac{x_n}{3} + \frac{5}{3x_n^2} = \frac{2x_n}{3} + \frac{5}{3x_n^2} = \frac{1}{3}\left(2x_n + \frac{5}{x_n^2}\right)
$$

Put $x_1 = 1$ the initial rough estimate of $\sqrt[3]{5}$. Then

$$x_1 = 1 \qquad x_2 = \frac{1}{3}\left(2 \cdot 1 + \frac{5}{1^2}\right) \approx 2.33333333333$$

$$x_2 = 2.33333333333 \qquad x_3 = \frac{1}{3}\left(2 \cdot 2.33333333333 + \frac{5}{2.33333333333^2}\right) \approx 1.86167800454$$

$$x_3 = 1.86167800454 \qquad x_4 = \frac{1}{3}\left(2 \cdot 1.86167800454 + \frac{5}{1.86167800454^2}\right) \approx 1.72200188006$$

$$x_4 = 1.72200188006 \qquad x_5 = \frac{1}{3}\left(2 \cdot 1.72200188006 + \frac{5}{1.72200188006^2}\right) \approx 1.71005973660$$

$$x_5 = 1.71005973660 \qquad x_6 = \frac{1}{3}\left(2 \cdot 1.71005973660 + \frac{5}{1.71005973660^2}\right) \approx 1.70997595078$$

$$x_6 = 1.70997595078 \qquad x_7 = \frac{1}{3}\left(2 \cdot 1.70997595078 + \frac{5}{1.70997595078^2}\right) \approx 1.70997594668$$

$$x_7 = 1.70997594668 \qquad x_8 = \frac{1}{3}\left(2 \cdot 1.70997594668 + \frac{5}{1.70997594668^2}\right) \approx 1.70997594668$$

EXAMPLE: Use Newton's Method to approximate the real solutions of $x^3 - x - 1 = 0$.

SOLUTION: We have

$$f(x) = x^3 - x - 1 \Longrightarrow f'(x) = 3x^2 - 1$$

so

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^3 - x_n - 1}{3x_n^2 - 1}$$

Put $x_1 = 1.5$ the initial rough estimate. Then

$$x_1 = 1.5 \qquad x_2 = 1.5 - \frac{1.5^3 - 1.5 - 1}{3 \cdot 1.5^2 - 1} \approx 1.34782609$$

$$x_2 = 1.34782609 \qquad x_3 = 1.34782609 - \frac{1.34782609^3 - 1.34782609 - 1}{3 \cdot 1.34782609^2 - 1} \approx 1.32520040$$

$$x_3 = 1.32520040 \qquad x_4 = 1.32520040 - \frac{1.32520040^3 - 1.32520040 - 1}{3 \cdot 1.32520040^2 - 1} \approx 1.32471817$$

$$x_4 = 1.32471817 \qquad x_5 = 1.32471817 - \frac{1.32471817^3 - 1.32471817 - 1}{3 \cdot 1.32471817^2 - 1} \approx 1.32471796$$

$$x_5 = 1.32471796 \qquad x_6 = 1.32471796 - \frac{1.32471796^3 - 1.32471796 - 1}{3 \cdot 1.32471796^2 - 1} \approx 1.32471796$$

Section 5.8 - Rolle's Theorem; Mean-Value Theorem

THEOREM (**Rolle's Theorem**): Let f be differentiable on (a, b) and continuous on $[a, b]$. If $f(a) = f(b) = 0$, then there is at least one number c in (a, b) such that $f'(c) = 0$.

THEOREM (**Mean-Value Theorem**): Let f be differentiable on (a, b) and continuous on $[a, b]$. Then there is at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

THEOREM (**The Constant Difference Theorem**): If f and g are continuous on a closed interval $[a, b]$, and if $f'(x) = g'(x)$ for all x in the open interval (a, b), then f and g differ by a constant on $[a, b]$; that is, there is a constant k such that $f(x) - g(x) = k$ for all x in $[a, b]$.