Section 5.5 - Absolute Maxima And Minima

DEFINITION: A function \( f \) is said to have an **absolute maximum** on an interval \( I \) at \( x_0 \) if \( f(x_0) \) is the largest value of \( f \) on \( I \); that is, \( f(x_0) \geq f(x) \) for all \( x \) in the domain of \( f \) that are in \( I \). Similarly, \( f \) is said to have an **absolute minimum** on an interval \( I \) at \( x_0 \) if \( f(x_0) \) is the smallest value of \( f \) on \( I \); that is, \( f(x_0) \leq f(x) \) for all \( x \) in the domain of \( f \) that are in \( I \). If \( f \) has either an absolute maximum or absolute minimum on \( I \) at \( x_0 \), then \( f \) is said to have an **absolute extremum** on \( I \) at \( x_0 \).

**THEOREM (Extreme-Value Theorem):** If a function \( f \) is continuous on a finite closed interval \([a, b]\), then \( f \) has both an absolute maximum and an absolute minimum on \([a, b]\).

**A Procedure for Finding the Absolute Extrema of a Continuous Function \( f \) on a Finite Closed Interval \([a, b]\).**

**Step 1:** Find the critical numbers of \( f \) in \((a, b)\).

**Step 2:** Evaluate \( f \) at all critical numbers and at the endpoints \( a \) and \( b \).

**Step 3:** The largest of the values in Step 2 is the absolute maximum value of \( f \) on \([a, b]\) and the smallest value is the absolute minimum.

**EXAMPLE:** Find the absolute maximum and minimum values of \( f(x) = 6x^{4/3} - 3x^{1/3} \) on the interval \([-1, 1]\), and determine where these values occur.

**SOLUTION:**

**Step 1:** Since \( f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}} \), there are 2 critical numbers \( x = 0 \) and \( x = \frac{1}{8} \).

**Step 2:** We now evaluate \( f \) at these critical numbers and at the endpoints \( x = -1 \) and \( x = 1 \). We have:

\[
\begin{align*}
    f(-1) &= 9, \\
    f(0) &= 0, \\
    f\left(\frac{1}{8}\right) &= -\frac{9}{8}, \\
    f(1) &= 3
\end{align*}
\]

**Step 3:** The largest value is 9 and the smallest value is \(-\frac{9}{8}\). Therefore an absolute maximum of \( f \) on \([-1, 1]\) is 9, occurring at \( x = -1 \) and an absolute minimum of \( f \) on \([-1, 1]\) is \(-\frac{9}{8}\), occurring at \( x = \frac{1}{8} \).

**EXAMPLE:** Find the absolute maximum and minimum values of \( f(x) = 2x^3 - 15x^2 + 36x \) on the interval \([1, 5]\), and determine where these values occur.

**SOLUTION:**

**Step 1:** Since \( f'(x) = 6x^2 - 30x + 36 = 6(x - 2)(x - 3) \), there are 2 critical numbers \( x = 2 \) and \( x = 3 \).

**Step 2:** We now evaluate \( f \) at these critical numbers and at the endpoints \( x = 1 \) and \( x = 5 \). We have:

\[
\begin{align*}
    f(1) &= 23, \\
    f(2) &= 28, \\
    f(3) &= 27, \\
    f(5) &= 55
\end{align*}
\]

**Step 3:** The largest value is 55 and the smallest value is 23. Therefore an absolute maximum of \( f \) on \([1, 5]\) is 55, occurring at \( x = 5 \) and an absolute minimum of \( f \) on \([1, 5]\) is 23, occurring at \( x = 1 \).
EXAMPLE: Determine whether \( f(x) = 3x^4 + 4x^3 \) has any absolute extrema.

SOLUTION:

Step 1: Since \( f'(x) = 12x^3 + 12x^2 = 12x^2(x + 1) \), there are 2 critical numbers \( x = 0 \) and \( x = -1 \).

Step 2: We now evaluate \( f \) at these critical numbers: \( f(0) = 0, f(-1) = -1 \). Also, we find the following limits:
\[
\lim_{x \to +\infty} f(x) = +\infty, \quad \lim_{x \to -\infty} f(x) = +\infty.
\]

Step 3: The smallest value is \(-1\). Therefore an absolute minimum of \( f \) on \((-\infty, +\infty)\) is \(-1\), occurring at \( x = -1 \).

EXAMPLE: Determine whether \( f(x) = \frac{1}{x^2 - x} \) has any absolute extrema on the interval \((0, 1)\). If so, find them and state where they occur.

SOLUTION:

Step 1: We have
\[
f'(x) = -\frac{2x - 1}{(x^2 - x)^2}.
\]
We see that \( f'(x) \) equals zero at \( x = \frac{1}{2} \) and does not exist at \( x = 0 \) and \( x = 1 \). But the last two points are not from \((0, 1)\). Therefore the only critical number is \( x = \frac{1}{2} \).

Step 2: We now evaluate \( f \) at this critical number:
\[
f\left(\frac{1}{2}\right) = -4
\]

Also, we find the following limits:
\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{1}{x^2 - x} = \lim_{x \to 0^+} \frac{1}{x(x - 1)} = -\infty,
\]
\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} \frac{1}{x^2 - x} = \lim_{x \to 1^-} \frac{1}{x(x - 1)} = -\infty.
\]

Step 3: The largest value is \(-4\). Therefore an absolute maximum of \( f \) on \((0, 1)\) is \(-4\), occurring at \( x = \frac{1}{2} \).

Section 5.6 - Applied Maximum And Minimum Problems

PROBLEM: A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

PROBLEM: An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

PROBLEM: Find the radius and height of the right circular cylinder of the largest volume that can be inscribed in a right circular cone with radius 6 inches and the height 10 inches.

PROBLEM: A closed cylindrical can is to hold 1 liter (1000 cm\(^3\)) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?