Section 5.5 - Absolute Maxima And Minima

DEFINITION: A function f is said to have an **absolute maximum** on an interval I at x_0 if $f(x_0)$ is the largest value of f on I; that is, $f(x_0) \geq f(x)$ for all x in the domain of f that are in I. Similarly, f is said to have an **absolute minimum** on an interval I at x_0 if $f(x_0)$ is the smallest value of f on I; that is, $f(x_0) \leq f(x)$ for all x in the domain of f that are in I. If f has either an absolute maximum or absolute minimum on I at x_0, then f is said to have an **absolute extremum** on I at x_0.

THEOREM (**Extreme-Value Theorem**): If a function f is continuous on a finite closed interval $[a, b]$, then f has both an absolute maximum and an absolute minimum on $[a, b]$.

A Procedure for Finding the Absolute Extrema of a Continuous Function f on a Finite Closed Interval $[a, b]$.

Step 1: Find the critical numbers of f in (a, b).

Step 2: Evaluate f at all critical numbers and at the endpoints a and b.

Step 3: The largest of the values in Step 2 is the absolute maximum value of f on $[a, b]$ and the smallest value is the absolute minimum.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = 6x^{4/3} - 3x^{1/3}$ on the interval $[-1, 1]$, and determine where these values occur.

SOLUTION:

Step 1: Since

$$f'(x) = 8x^{1/3} - x^{-2/3} = x^{-2/3}(8x - 1) = \frac{8x - 1}{x^{2/3}},$$

there are 2 critical numbers $x = 0$ and $x = \frac{1}{8}$.

Step 2: We now evaluate f at these critical numbers and at the endpoints $x = -1$ and $x = 1$. We have:

$$f(-1) = 9, \quad f(0) = 0, \quad f\left(\frac{1}{8}\right) = -\frac{9}{8}, \quad f(1) = 3$$

Step 3: The largest value is 9 and the smallest value is $-\frac{9}{8}$. Therefore an absolute maximum of f on $[-1, 1]$ is 9, occurring at $x = -1$ and an absolute minimum of f on $[-1, 1]$ is $-\frac{9}{8}$, occurring at $x = \frac{1}{8}$.

EXAMPLE: Find the absolute maximum and minimum values of $f(x) = 2x^3 - 15x^2 + 36x$ on the interval $[1, 5]$, and determine where these values occur.

SOLUTION:

Step 1: Since $f'(x) = 6x^2 - 30x + 36 = 6(x - 2)(x - 3)$, there are 2 critical numbers $x = 2$ and $x = 3$.

Step 2: We now evaluate f at these critical numbers and at the endpoints $x = 1$ and $x = 5$. We have:

$$f(1) = 23, \quad f(2) = 28, \quad f(3) = 27, \quad f(5) = 55$$

Step 3: The largest value is 55 and the smallest value is 23. Therefore an absolute maximum of f on $[1, 5]$ is 55, occurring at $x = 5$ and an absolute minimum of f on $[1, 5]$ is 23, occurring at $x = 1$.

EXAMPLE: Determine whether $f(x) = 3x^4 + 4x^3$ has any absolute extrema.

SOLUTION:

Step 1: Since $f'(x) = 12x^3 + 12x^2 = 12x^2(x + 1)$, there are 2 critical numbers $x = 0$ and $x = -1$.

Step 2: We now evaluate f at these critical numbers: $f(0) = 0$, $f(-1) = -1$. Also, we find the following limits:

$$\lim_{x\to+\infty} f(x) = +\infty, \qquad \lim_{x\to-\infty} f(x) = +\infty.$$

Step 3: The smallest value is -1. Therefore an absolute minimum of f on $(-\infty, +\infty)$ is -1, occurring at $x = -1$.

EXAMPLE: Determine whether $f(x) = \dfrac{1}{x^2 - x}$ has any absolute extrema on the interval $(0, 1)$. If so, find them and state where they occur.

SOLUTION:

Step 1: We have

$$f'(x) = -\frac{2x - 1}{(x^2 - x)^2}.$$

We see that $f'(x)$ equals zero at $x = \frac{1}{2}$ and does not exist at $x = 0$ and $x = 1$. But the last two points are not from $(0, 1)$. Therefore the only critical number is $x = \frac{1}{2}$.

Step 2: We now evaluate f at this critical number:

$$f\left(\frac{1}{2}\right) = -4$$

Also, we find the following limits:

$$\lim_{x\to0^+} f(x) = \lim_{x\to0^+} \frac{1}{x^2 - x} = \lim_{x\to0^+} \frac{1}{x(x-1)} = -\infty,$$

$$\lim_{x\to1^-} f(x) = \lim_{x\to1^-} \frac{1}{x^2 - x} = \lim_{x\to1^-} \frac{1}{x(x-1)} = -\infty.$$

Step 3: The largest value is -4. Therefore an absolute maximum of f on $(0, 1)$ is -4, occurring at $x = \frac{1}{2}$.

Section 5.6 - Applied Maximum And Minimum Problems

PROBLEM: A garden is to be laid out in a rectangular area and protected by a chicken wire fence. What is the largest possible area of the garden if only 100 running feet of chicken wire is available for the fence?

PROBLEM: An open box is to be made from a 16-inch by 30-inch piece of cardboard by cutting out squares of equal size from the four corners and bending up the sides. What size should the squares be to obtain a box with the largest volume?

PROBLEM: Find the radius and height of the right circular cylinder of the largest volume that can be inscribed in a right circular cone with radius 6 inches and the height 10 inches.

PROBLEM: A closed cylindrical can is to hold 1 liter (1000 cm^3) of liquid. How should we choose the height and radius to minimize the amount of material needed to manufacture the can?