

EXAMPLE: Generate or sketch a graph of the equation

$$y = \frac{x^2 - 1}{x^3}$$

and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries*: Replacing x by $-x$ and y by $-y$ yields an equation that simplifies back to the original equation, so the graph is symmetric about the origin.

(b) *x-intercepts*: Setting $y = 0$ yields the x -intercepts $x = -1$ and $x = 1$.

(c) *y-intercepts*: Setting $x = 0$ leads to a division by zero, so that there is no y -intercept.

(d) *Vertical asymptotes*: Setting $x^3 = 0$ yields the solutions $x = 0$. This is not a root of $x^2 - 1$, so the graph has a vertical asymptote at $x = 0$.

(e) *Horizontal asymptotes:* The limits

$$\lim_{x \rightarrow +\infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow +\infty} \frac{1}{x} - \frac{1}{x^3} = 0$$

$$\lim_{x \rightarrow -\infty} \frac{x^2 - 1}{x^3} = \lim_{x \rightarrow -\infty} \frac{1}{x} - \frac{1}{x^3} = 0$$

yield the horizontal asymptote $y = 0$.

(f) *Derivatives:*

$$\frac{dy}{dx} = \frac{(2x)x^3 - (x^2 - 1)(3x^2)}{(x^3)^2} = \frac{3 - x^2}{x^4}$$
$$\frac{d^2y}{dx^2} = \frac{(-2x)x^4 - (3 - x^2)(4x^3)}{(x^4)^2} = \frac{2(x^2 - 6)}{x^5}$$

(g) *Interval of increase and decrease; relative extrema:* There are 3 critical numbers: $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $[-\sqrt{3}, 0)$ and $(0, \sqrt{3}]$; and it is decreasing on the intervals $(-\infty, -\sqrt{3}]$ and $[\sqrt{3}, +\infty)$. There is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.

(e) *Concavity:* There are 3 points at which the second derivative does not exist or is equal to

zero: $x = 0$, $x = -\sqrt{6}$ and $x = \sqrt{6}$. A sign analysis yields: the graph is concave up on the intervals $[-\sqrt{6}, 0)$ and $[\sqrt{6}, +\infty)$; it is concave down on $(-\infty, -\sqrt{6}]$ and $(0, \sqrt{6}]$. There are 2 inflection points: $x = -\sqrt{6}$ and $x = \sqrt{6}$.

EXAMPLE: Generate or sketch a graph of the equation

$$y = \frac{x^3 - x^2 - 8}{x - 1}$$

and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries*: There are no symmetries about a vertical axis or about a point.

(b) *x-intercepts*: Setting $y = 0$ leads to solving the equation $x^3 - x^2 - 8 = 0$. There is one solution $x \approx 2.39486$.

(c) *y-intercepts*: Setting $x = 0$ yields to the y -intercept $y = 8$.

(d) *Vertical asymptotes*: Setting $x = 1$ would produce a nonzero numerator and a zero denominator for $f(x)$, so the graph has a vertical asymptote at $x = 1$.

(e) *Horizontal asymptotes*: There are no hor-

horizontal asymptotes, since

$$\lim_{x \rightarrow +\infty} \frac{x^3 - x^2 - 8}{x - 1} = \lim_{x \rightarrow +\infty} \frac{1 - 1/x - 8/x^3}{1/x^2 - 1/x^3} = +\infty$$

$$\lim_{x \rightarrow -\infty} \frac{x^3 - x^2 - 8}{x - 1} = \lim_{x \rightarrow -\infty} \frac{1 - 1/x - 8/x^3}{1/x^2 - 1/x^3} = +\infty$$

However, since

$$f(x) = x^2 - \frac{8}{x - 1},$$

we have

$$\lim_{x \rightarrow +\infty} [f(x) - x^2] = \lim_{x \rightarrow +\infty} -\frac{8}{x - 1} = 0$$

$$\lim_{x \rightarrow -\infty} [f(x) - x^2] = \lim_{x \rightarrow -\infty} -\frac{8}{x - 1} = 0$$

therefore $f(x)$ is asymptotic to $y = x^2$ and $x \rightarrow -\infty$ and as $x \rightarrow +\infty$.

(f) *Derivatives:*

$$\frac{dy}{dx} = \frac{(3x^2 - 2x)(x - 1) - (x^3 - x^2 - 8) \cdot 1}{(x - 1)^2}$$

$$= \frac{2(x^3 - 2x^2 + x + 4)}{(x - 1)^2}$$

$$= \frac{2(x + 1)(x^2 - 3x + 4)}{(x - 1)^2}$$

$$\frac{d^2y}{dx^2} = \frac{2(x^3 - 3x^2 + 3x - 9)}{(x - 1)^3}$$

(g) *Interval of increase and decrease; relative extrema:* There are 2 critical numbers: $x = -1$, $x = 1$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $[-1, 1)$ and $(1, +\infty)$; and it is decreasing on the interval $(-\infty, -1]$. There is a relative minimum at $x = -1$.

(e) *Concavity:* There are 2 points at which the second derivative does not exist or is equal to zero: $x = 1$ and $x = 3$. A sign analysis yields: the graph is concave up on the intervals $(-\infty, 1)$ and $[3, +\infty)$; it is concave down on $(1, 3]$. There

is 1 inflection point: $x = 3$.

EXAMPLE: Generate or sketch a graph of the equation

$$y = \frac{2x^2 - 8}{x^2 - 16}$$

and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries*: Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y -axis.

(b) *x -intercepts*: Setting $y = 0$ yields the x -intercepts $x = -2$ and $x = 2$.

(c) *y -intercepts*: Setting $x = 0$ yields the y -intercepts $y = 1/2$.

(d) *Vertical asymptotes*: Setting $x^2 - 16 = 0$ yields the solutions $x = -4$ and $x = 4$. Since neither solution is a root of $2x^2 - 8$, the graph has vertical asymptotes at $x = -4$ and $x = 4$.

(e) *Horizontal asymptotes:* The limits

$$\lim_{x \rightarrow +\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x \rightarrow +\infty} \frac{2 - 8/x^2}{1 - 16/x^2} = 2$$

$$\lim_{x \rightarrow -\infty} \frac{2x^2 - 8}{x^2 - 16} = \lim_{x \rightarrow -\infty} \frac{2 - 8/x^2}{1 - 16/x^2} = 2$$

yield the horizontal asymptote $y = 2$.

(f) *Derivatives:*

$$\frac{dy}{dx} = \frac{(4x)(x^2 - 16) - (2x^2 - 8)(2x)}{(x^2 - 16)^2}$$

$$= -\frac{48x}{(x^2 - 16)^2}$$

$$\frac{d^2y}{dx^2} = \frac{48(16 + 3x^2)}{(x^2 - 16)^3}$$

(g) *Interval of increase and decrease; relative extrema:* There are 3 critical numbers: $x = 0$, $x = -4$, $x = 4$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $(-\infty, -4)$ and $(-4, 0]$; and it is decreasing on the intervals $[0, 4)$ and $(4, +\infty)$. There is a relative maximum at $x = 0$.

(e) *Concavity*: There are 2 points at which the second derivative does not exist: $x = -4$ and $x = 4$. A sign analysis yields: the graph is concave up on the intervals $(-\infty, -4)$ and $(4, +\infty)$; it is concave down on $(-4, 4)$. There are no inflection points, since $f''(x)$ does not exist at $x = -4$ and $x = 4$.