Section 5.3 - Analysis Of Functions III: Applying Technology And The Tools Of Calculus

EXAMPLE: Generate or sketch a graph of the equation $y = \dfrac{x^2-1}{x^3}$ and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries:* Replacing x by $-x$ and y by $-y$ yields an equation that simplifies back to the original equation, so the graph is symmetric about the origin.

(b) *x-intercepts:* Setting $y = 0$ yields the x-intercepts $x = -1$ and $x = 1$.

(c) *y-intercepts:* Setting $x = 0$ leads to a division by zero, so that there is no y-intercept.

(d) *Vertical asymptotes:* Setting $x^3 = 0$ yields the solutions $x = 0$. This is not a root of $x^2 - 1$, so the graph has a vertical asymptote at $x = 0$.

(e) *Horizontal asymptotes:* The limits

$$\lim_{x\to+\infty} \frac{x^2-1}{x^3} = \lim_{x\to+\infty} \frac{1}{x} - \frac{1}{x^3} = 0, \quad \lim_{x\to-\infty} \frac{x^2-1}{x^3} = \lim_{x\to-\infty} \frac{1}{x} - \frac{1}{x^3} = 0$$

yield the horizontal asymptote $y = 0$.

(f) *Derivatives:*

$$\frac{dy}{dx} = \frac{(2x)x^3 - (x^2-1)(3x^2)}{(x^3)^2} = \frac{3-x^2}{x^4}$$

$$\frac{d^2y}{dx^2} = \frac{(-2x)x^4 - (3-x^2)(4x^3)}{(x^4)^2} = \frac{2(x^2-6)}{x^5}$$

(g) *Interval of increase and decrease; relative extrema:* There are 3 critical numbers: $x = 0$, $x = -\sqrt{3}$, $x = \sqrt{3}$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $[-\sqrt{3}, 0)$ and $(0, \sqrt{3}]$; and it is decreasing on the intervals $(-\infty, -\sqrt{3}]$ and $[\sqrt{3}, +\infty)$. There is a relative minimum at $x = -\sqrt{3}$ and a relative maximum at $x = \sqrt{3}$.

(e) *Concavity:* There are 3 points at which the second derivative does not exist or is equal to zero: $x = 0$, $x = -\sqrt{6}$ and $x = \sqrt{6}$. A sign analysis yields: the graph is concave up on the intervals $[-\sqrt{6}, 0)$ and $[\sqrt{6}, +\infty)$; it is concave down on $(-\infty, -\sqrt{6}]$ and $(0, \sqrt{6}]$. There are 2 inflection points: $x = -\sqrt{6}$ and $x = \sqrt{6}$.

EXAMPLE: Generate or sketch a graph of the equation $y = \dfrac{x^3 - x^2 - 8}{x-1}$ and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries:* There are no symmetries about a vertical axis or about a point.

(b) *x-intercepts:* Setting $y = 0$ leads to solving the equation $x^3 - x^2 - 8 = 0$. There is one solution $x \approx 2.39486$.

(c) *y-intercepts:* Setting $x = 0$ yields to the y-intercept $y = 8$.

(d) *Vertical asymptotes:* Setting $x = 1$ would produce a nonzero numerator and a zero denominator for $f(x)$, so the graph has a vertical asymptote at $x = 1$.

(e) *Horizontal asymptotes:* There are no horizontal asymptotes, since

$$\lim_{x\to+\infty} \frac{x^3-x^2-8}{x-1} = \lim_{x\to+\infty} \frac{1-1/x-8/x^3}{1/x^2-1/x^3} = +\infty, \quad \lim_{x\to-\infty} \frac{x^3-x^2-8}{x-1} = \lim_{x\to-\infty} \frac{1-1/x-8/x^3}{1/x^2-1/x^3} = +\infty$$

However, since

$$f(x) = x^2 - \frac{8}{x-1},$$

we have

$$\lim_{x\to+\infty}[f(x)-x^2]=\lim_{x\to+\infty}-\frac{8}{x-1}=0,\quad \lim_{x\to-\infty}[f(x)-x^2]=\lim_{x\to-\infty}-\frac{8}{x-1}=0$$

therefore $f(x)$ is asymptotic to $y=x^2$ and $x\to-\infty$ and as $x\to+\infty$.

(f) *Derivatives:*

$$\frac{dy}{dx}=\frac{(3x^2-2x)(x-1)-(x^3-x^2-8)\cdot 1}{(x-1)^2}=\frac{2(x^3-2x^2+x+4)}{(x-1)^2}=\frac{2(x+1)(x^2-3x+4)}{(x-1)^2}$$

$$\frac{d^2y}{dx^2}=\frac{2(x^3-3x^2+3x-9)}{(x-1)^3}$$

(g) *Interval of increase and decrease; relative extrema:* There are 2 critical numbers: $x=-1,\ x=1$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $[-1,1)$ and $(1,+\infty)$; and it is decreasing on the interval $(-\infty,-1]$. There is a relative minimum at $x=-1$.

(e) *Concavity:* There are 2 points at which the second derivative does not exist or is equal to zero: $x=1$ and $x=3$. A sign analysis yields: the graph is concave up on the intervals $(-\infty,1)$ and $[3,+\infty)$; it is concave down on $(1,3]$. There is 1 inflection point: $x=3$.

EXAMPLE: Generate or sketch a graph of the equation $y=\dfrac{2x^2-8}{x^2-16}$ and identify the exact locations of the intercepts, relative extrema, inflection points, and asymptotes.

SOLUTION:

(a) *Symmetries:* Replacing x by $-x$ does not change the equation, so the graph is symmetric about the y-axis.

(b) *x-intercepts:* Setting $y=0$ yields the x-intercepts $x=-2$ and $x=2$.

(c) *y-intercepts:* Setting $x=0$ yields the y-intercepts $y=1/2$.

(d) *Vertical asymptotes:* Setting $x^2-16=0$ yields the solutions $x=-4$ and $x=4$. Since neither solution is a root of $2x^2-8$, the graph has vertical asymptotes at $x=-4$ and $x=4$.

(e) *Horizontal asymptotes:* The limits

$$\lim_{x\to+\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to+\infty}\frac{2-8/x^2}{1-16/x^2}=2,\quad \lim_{x\to-\infty}\frac{2x^2-8}{x^2-16}=\lim_{x\to-\infty}\frac{2-8/x^2}{1-16/x^2}=2$$

yield the horizontal asymptote $y=2$.

(f) *Derivatives:*

$$\frac{dy}{dx}=\frac{(4x)(x^2-16)-(2x^2-8)(2x)}{(x^2-16)^2}=-\frac{48x}{(x^2-16)^2}$$

$$\frac{d^2y}{dx^2}=\frac{48(16+3x^2)}{(x^2-16)^3}$$

(g) *Interval of increase and decrease; relative extrema:* There are 3 critical numbers: $x=0,\ x=-4,\ x=4$. A sign analysis of dy/dx yields: the graph is increasing on the intervals $(-\infty,-4)$ and $(-4,0]$; and it is decreasing on the intervals $[0,4)$ and $(4,+\infty)$. There is a relative maximum at $x=0$.

(e) *Concavity:* There are 2 points at which the second derivative does not exist: $x=-4$ and $x=4$. A sign analysis yields: the graph is concave up on the intervals $(-\infty,-4)$ and $(4,+\infty)$; it is concave down on $(-4,4)$. There are no inflection points, since $f(x)$ does not exist at $x=-4$ and $x=4$.