

## Section 5.2 - Analysis Of Functions II: Relative Extrema; First And Second Derivative Tests

DEFINITION: A function  $f$  is said to have a **relative maximum** at  $x_0$  if there is an open interval containing  $x_0$  on which  $f(x_0)$  is the largest value, that is,  $f(x_0) \geq f(x)$  for all  $x$  in the interval. Similarly,  $f$  is said to have a **relative minimum** at  $x_0$  if there is an open interval containing  $x_0$  on which  $f(x_0)$  is the smallest value, that is,  $f(x_0) \leq f(x)$  for all  $x$  in the interval. If  $f$  has either a relative maximum or a relative minimum at  $x_0$ , then  $f$  is said to have a **relative extremum** at  $x_0$ .

THEOREM: Suppose that  $f$  is a function defined on an open interval containing the number  $x_0$ . If  $f$  has a relative extremum at  $x = x_0$ , then either  $f'(x_0) = 0$  or  $f$  is not differentiable at  $x_0$ .

DEFINITION: Values in the domain of  $f$  at which either  $f'(x) = 0$  or  $f$  is not differentiable are called **critical numbers** of  $f$ .

THEOREM' : If a function  $f$  is defined on an open interval, its relative extrema on the interval, if any, occur at critical numbers.

THEOREM (**The First Derivative Test**): Suppose  $f$  is continuous at a critical number  $x_0$ .

(a) If  $f'(x) > 0$  on an open interval extending left from  $x_0$  and  $f'(x) < 0$  on an open interval extending right from  $x_0$ , then  $f$  has a relative maximum at  $x_0$ .

(b) If  $f'(x) < 0$  on an open interval extending left from  $x_0$  and  $f'(x) > 0$  on an open interval extending right from  $x_0$ , then  $f$  has a relative minimum at  $x_0$ .

(c) If  $f'(x)$  has the same sign [either  $f'(x) > 0$  or  $f'(x) < 0$ ] on an open interval extending left from  $x_0$  and on an open interval extending right from  $x_0$ , then  $f$  does not have a relative extremum at  $x_0$ .

THEOREM (**The Second Derivative Test**): Suppose that  $f$  is twice differentiable at  $x_0$ .

(a) If  $f'(x_0) = 0$  and  $f''(x_0) > 0$ , then  $f$  has a relative minimum at  $x_0$ .

(b) If  $f'(x_0) = 0$  and  $f''(x_0) < 0$ , then  $f$  has a relative maximum at  $x_0$ .

(c) If  $f'(x_0) = 0$  and  $f''(x_0) = 0$ , then the test is inconclusive; that is,  $f$  may have a relative maximum, a relative minimum, or neither at  $x_0$ .