

DEFINITION: The **inverse sine function**, denoted by  $\sin^{-1}$ , is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$$

DEFINITION: The **inverse cosine function**, denoted by  $\cos^{-1}$ , is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \leq x \leq \pi$$

DEFINITION: The **inverse tangent function**, denoted by  $\tan^{-1}$ , is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

DEFINITION: The **inverse secant function**, denoted by  $\sec^{-1}$ , is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \leq x \leq \pi \quad \text{with} \quad x \neq \frac{\pi}{2}$$

IMPORTANT: Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\sec x}$$

EXAMPLE:

$$\sin^{-1} 0 = 0, \quad \sin^{-1} 1 = \frac{\pi}{2}, \quad \sin^{-1}(-1) = -\frac{\pi}{2}$$

$$\sin^{-1} \frac{1}{2} = \frac{\pi}{6}, \quad \sin^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{3}, \quad \sin^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

EXAMPLE:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi$$

$$\cos^{-1} \frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1} \frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1} \frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

EXAMPLE:

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

$$\tan^{-1} \frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1} \left( -\frac{1}{\sqrt{3}} \right) = -\frac{\pi}{6}$$



FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$

THEOREM: We have

$$(a) \quad (\sin^{-1} u)' = \frac{1}{\sqrt{1-u^2}} u', \quad -1 < u < 1$$

$$(b) \quad (\cos^{-1} u)' = -\frac{1}{\sqrt{1-u^2}} u', \quad -1 < u < 1$$

$$(c) \quad (\tan^{-1} u)' = \frac{1}{1+u^2} u', \quad -\infty < u < +\infty$$

$$(d) \quad (\sec^{-1} u)' = \frac{1}{|u|\sqrt{u^2-1}} u', \quad |u| > 1$$

(a) Let  $y = \sin^{-1} u$ , then  $\sin y = u$ . Therefore

$$(\sin y)' = u' \implies \cos y \cdot y' = u'$$

so

$$y' = \frac{u'}{\cos y}$$

Note, that

$$\cos y = \sqrt{1 - \sin^2 y} = [\sin y = u] = \sqrt{1 - u^2}.$$

Hence

$$y' = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1 - u^2}}$$

(b) Let  $y = \cos^{-1} u$ , then  $\cos y = u$ . Therefore

$$(\cos y)' = u' \implies -\sin y \cdot y' = u'$$

so

$$y' = -\frac{u'}{\sin y}$$

Note, that

$$\sin y = \sqrt{1 - \cos^2 y} = [\cos y = u] = \sqrt{1 - u^2}.$$

Hence

$$y' = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1 - u^2}}$$

(c) Let  $y = \tan^{-1} u$ , then  $\tan y = u$ . Therefore

$$(\tan y)' = u' \implies \sec^2 y \cdot y' = u'$$

so

$$y' = \frac{u'}{\sec^2 y}$$

Note, that

$$\sec^2 y = 1 + \tan^2 y = [\tan y = u] = 1 + u^2.$$

Hence

$$y' = \frac{u'}{\sec^2 y} = \frac{u'}{1 + u^2}$$

(d) Let  $y = \sec^{-1} u$ , then  $\sec y = u$ . Therefore

$$(\sec y)' = u' \implies \sec y \tan y \cdot y' = u'$$

so

$$y' = \frac{u'}{\sec y \tan y}$$

Note, that

$$\begin{aligned} \sec y \tan y &= |\sec y| \sqrt{\sec^2 y - 1} = [\sec y = u] \\ &= |u| \sqrt{u^2 - 1}. \end{aligned}$$

Hence

$$y' = \frac{u'}{\sec y \tan y} = \frac{u'}{|u| \sqrt{u^2 - 1}}$$