Section 4.4 - Inverse Trigonometric Functions And Their Derivatives

DEFINITION: The **inverse sine function,** denoted by $\sin^{-1}$, is defined to be the inverse of the restricted sine function

$$\sin x, \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

DEFINITION: The **inverse cosine function,** denoted by $\cos^{-1}$, is defined to be the inverse of the restricted cosine function

$$\cos x, \quad 0 \le x \le \pi$$

DEFINITION: The **inverse tangent function,** denoted by $\tan^{-1}$, is defined to be the inverse of the restricted tangent function

$$\tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$$

DEFINITION: The **inverse secant function,** denoted by $\sec^{-1}$, is defined to be the inverse of the restricted secant function

$$\sec x, \quad 0 \le x \le \pi \quad \text{with} \quad x \ne \frac{\pi}{2}$$

<u>IMPORTANT:</u> Do not confuse

$$\sin^{-1} x, \quad \cos^{-1} x, \quad \tan^{-1} x, \quad \sec^{-1} x$$

with

$$\frac{1}{\sin x}, \quad \frac{1}{\cos x}, \quad \frac{1}{\tan x}, \quad \frac{1}{\sec x}$$

EXAMPLE:

$$\sin^{-1} 0 = 0, \quad \sin^{-1} 1 = \frac{\pi}{2}, \quad \sin^{-1}(-1) = -\frac{\pi}{2}, \quad \sin^{-1}\frac{1}{2} = \frac{\pi}{6}, \quad \sin^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{3}, \quad \sin^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

EXAMPLE:

$$\cos^{-1} 0 = \frac{\pi}{2}, \quad \cos^{-1} 1 = 0, \quad \cos^{-1}(-1) = \pi, \quad \cos^{-1}\frac{1}{2} = \frac{\pi}{3}, \quad \cos^{-1}\frac{\sqrt{3}}{2} = \frac{\pi}{6}, \quad \cos^{-1}\frac{\sqrt{2}}{2} = \frac{\pi}{4}$$

EXAMPLE:

$$\tan^{-1} 1 = \frac{\pi}{4}, \quad \tan^{-1}(-1) = -\frac{\pi}{4}, \quad \tan^{-1}\sqrt{3} = \frac{\pi}{3}, \quad \tan^{-1}\frac{1}{\sqrt{3}} = \frac{\pi}{6}, \quad \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right) = -\frac{\pi}{6}$$

FUNCTION	DOMAIN	RANGE
$\sin^{-1} x$	$[-1, 1]$	$[-\pi/2, \pi/2]$
$\cos^{-1} x$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1} x$	$(-\infty, +\infty)$	$(-\pi/2, \pi/2)$
$\sec^{-1} x$	$(-\infty, -1] \cup [1, +\infty)$	$[0, \pi/2) \cup (\pi/2, \pi]$

THEOREM: We have

(a) $(\sin^{-1} u)' = \dfrac{1}{\sqrt{1-u^2}} u', \quad -1 < u < 1$

(b) $(\cos^{-1} u)' = -\dfrac{1}{\sqrt{1-u^2}} u', \quad -1 < u < 1$

(c) $(\tan^{-1} u)' = \dfrac{1}{1+u^2} u', \quad -\infty < u < +\infty$

(d) $(\sec^{-1} u)' = \dfrac{1}{|u|\sqrt{u^2-1}} u', \quad |u| > 1$

PROOF:

(a) Let $y = \sin^{-1} u$, then $\sin y = u$. Therefore

$$(\sin y)' = u' \quad \Longrightarrow \quad \cos y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\cos y}$$

Note, that $\cos y = \sqrt{1 - \sin^2 y} = [\sin y = u] = \sqrt{1-u^2}$. Hence

$$y' = \frac{u'}{\cos y} = \frac{u'}{\sqrt{1-u^2}}$$

(b) Let $y = \cos^{-1} u$, then $\cos y = u$. Therefore

$$(\cos y)' = u' \quad \Longrightarrow \quad -\sin y \cdot y' = u' \quad \Longrightarrow \quad y' = -\frac{u'}{\sin y}$$

Note, that $\sin y = \sqrt{1 - \cos^2 y} = [\cos y = u] = \sqrt{1-u^2}$. Hence

$$y' = -\frac{u'}{\sin y} = -\frac{u'}{\sqrt{1-u^2}}$$

(c) Let $y = \tan^{-1} u$, then $\tan y = u$. Therefore

$$(\tan y)' = u' \quad \Longrightarrow \quad \sec^2 y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec^2 y}$$

Note, that $\sec^2 y = 1 + \tan^2 y = [\tan y = u] = 1 + u^2$. Hence

$$y' = \frac{u'}{\sec^2 y} = \frac{u'}{1+u^2}$$

(d) Let $y = \sec^{-1} u$, then $\sec y = u$. Therefore

$$(\sec y)' = u' \quad \Longrightarrow \quad \sec y \tan y \cdot y' = u' \quad \Longrightarrow \quad y' = \frac{u'}{\sec y \tan y}$$

Note, that $\sec y \tan y = |\sec y| \sqrt{\sec^2 y - 1} = [\sec y = u] = |u|\sqrt{u^2-1}$. Hence

$$y' = \frac{u'}{\sec y \tan y} = \frac{u'}{|u|\sqrt{u^2-1}}$$