

THEOREM: We have

$$\begin{aligned}\lim_{x \rightarrow +\infty} \left(1 + \frac{1}{x}\right)^x &= \lim_{x \rightarrow -\infty} \left(1 + \frac{1}{x}\right)^x \\ &= \lim_{x \rightarrow 0} (1 + x)^{1/x} \\ &= e \approx 2.718282\end{aligned}$$

NOTATION: $\log_e x = \ln x$

IMPORTANT FORMULA: We have

$$\log_b x = \frac{\log_a x}{\log_a b}$$

In particular, if $a = e$, then

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PROOF: By the definition of the logarithmic function we have $b^{\log_b x} = x$, hence

$$\log_a \left(b^{\log_b x} \right) = \log_a x,$$

which implies

$$\log_b x \log_a b = \log_a x,$$

and the result follows. ■

THEOREM: We have

$$\frac{d}{dx}[\log_b x] = \frac{1}{x \ln b}$$

PROOF: We have:

$$\begin{aligned} \frac{d}{dx}[\log_b x] &= \lim_{h \rightarrow 0} \frac{\log_b(x+h) - \log_b x}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{h} \log_b \left(\frac{x+h}{x} \right) \right] \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{h} \log_b \left(1 + \frac{h}{x} \right) \right] \\ &= \lim_{h \rightarrow 0} \left[\frac{1}{x} \frac{x}{h} \log_b \left(1 + \frac{h}{x} \right) \right] \\ &= \frac{1}{x} \lim_{h \rightarrow 0} \left[\log_b \left(1 + \frac{h}{x} \right)^{x/h} \right] \\ &= \frac{1}{x} \lim_{h \rightarrow 0} \left[\log_b \left(1 + \frac{h}{x} \right)^{1/(h/x)} \right] \\ &= \frac{1}{x} \log_b e \\ &= \frac{1 \ln e}{x \ln b} \\ &= \frac{1}{x \ln b} \end{aligned}$$



COROLLARY: We have

$$\frac{d}{dx}[\ln x] = \frac{1}{x}$$

REMARK: In general,

$$(\log_b u)' = \frac{1}{u \ln b} \cdot u' \quad \text{and} \quad (\ln u)' = \frac{1}{u} \cdot u',$$

$$[\ln(x^2 + 1)]' = \frac{1}{x^2 + 1} \cdot (x^2 + 1)' = \frac{2x}{x^2 + 1}$$

$$[\ln(\ln x)]' = \frac{1}{\ln x} \cdot (\ln x)' = \frac{1}{x \ln x}$$

$$\begin{aligned}\left[\ln\left(\frac{x}{1+x^2}\right)\right]' &= \frac{1+x^2}{x} \cdot \left(\frac{x}{1+x^2}\right)' \\ &= \frac{1+x^2}{x} \cdot \frac{x'(1+x^2) - x(1+x^2)'}{(1+x^2)^2} \\ &= \frac{1+x^2}{x} \cdot \frac{1-x^2}{(1+x^2)^2}\end{aligned}$$

Find the derivative of

$$y = \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4}$$

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SOLUTION: We have

$$\begin{aligned} \ln y &= \ln \left(\frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4} \right) \\ &= 2 \ln x + \frac{1}{3} \ln(7x - 14) - 4 \ln(1 + x^2), \end{aligned}$$

therefore

$$(\ln y)' = \left(2 \ln x + \frac{1}{3} \ln(7x - 14) - 4 \ln(1 + x^2) \right)',$$

so

$$\frac{1}{y} \cdot y' = \frac{2}{x} + \frac{7}{3(7x - 14)} - 4 \frac{2x}{1 + x^2} = \frac{2}{x} + \frac{1}{3x - 6} - \frac{8x}{1 + x^2}.$$

It follows that

$$\begin{aligned} y' &= y \left(\frac{2}{x} + \frac{1}{3x - 6} - \frac{8x}{1 + x^2} \right) \\ &= \frac{x^2 \sqrt[3]{7x - 14}}{(1 + x^2)^4} \left(\frac{2}{x} + \frac{1}{3x - 6} - \frac{8x}{1 + x^2} \right). \end{aligned}$$

EXAMPLE: Find the derivative of

$$y = \frac{(x^2 - 8)^{1/3} \sqrt{x^3 + 1}}{x^6 - 7x + 5}$$

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SOLUTION: We have

$$\begin{aligned} \ln y &= \ln \left(\frac{(x^2 - 8)^{1/3} \sqrt{x^3 + 1}}{x^6 - 7x + 5} \right) \\ &= \frac{1}{3} \ln(x^2 - 8) + \frac{1}{2} \ln(x^3 + 1) - \ln(x^6 - 7x + 5), \end{aligned}$$

therefore

$$(\ln y)' = \left(\frac{1}{3} \ln(x^2 - 8) + \frac{1}{2} \ln(x^3 + 1) - \ln(x^6 - 7x + 5) \right)',$$

so

$$\frac{1}{y} \cdot y' = \frac{2x}{3(x^2 - 8)} + \frac{3x^2}{2(x^3 + 1)} - \frac{6x^5 - 7}{x^6 - 7x + 5}.$$

It follows that

$$\begin{aligned} y' &= y \left(\frac{2x}{3(x^2 - 8)} + \frac{3x^2}{2(x^3 + 1)} - \frac{6x^5 - 7}{x^6 - 7x + 5} \right) \\ &= \frac{(x^2 - 8)^{1/3} \sqrt{x^3 + 1}}{x^6 - 7x + 5} \left(\frac{2x}{3(x^2 - 8)} + \frac{3x^2}{2(x^3 + 1)} - \frac{6x^5 - 7}{x^6 - 7x + 5} \right) \end{aligned}$$

THEOREM: We have

$$\frac{d}{dx}[b^x] = b^x \ln b$$

In particular, if $b = e$, then

$$(e^x)' = e^x.$$

PROOF: Let $y = b^x$. We should prove that $y' = b^x \ln b$. We have:

$$y = b^x \quad \Longrightarrow \quad \ln y = x \ln b,$$

therefore

$$(\ln y)' = (x \ln b)' \quad \Longrightarrow \quad \frac{1}{y} \cdot y' = \ln b,$$

so

$$y' = y \ln b = b^x \ln b. \quad \blacksquare$$

REMARK: In general,

$$(b^u)' = b^u \ln b \cdot u' \quad \text{and} \quad (e^u)' = e^u \cdot u'.$$

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EXAMPLE:

$$(2^{\sin 2x})' = 2^{\sin 2x} \ln 2 \cdot (\sin 2x)' = 2^{\sin 2x} \ln 2 \cdot \cos 2x \cdot 2$$