Section 4.3 - Derivatives of Logarithmic and Exponential Functions

THEOREM: We have

$$\lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to-\infty}\left(1+\frac{1}{x}\right)^x=\lim_{x\to 0}(1+x)^{1/x}=e\approx 2.718282$$

NOTATION: $\log_e x=\ln x$

<u>IMPORTANT FORMULA</u>: We have

$$\log_b x=\frac{\log_a x}{\log_a b}$$

In particularly, if $a=e$, then

$$\log_b x=\frac{\ln x}{\ln b}$$

PROOF: By the definition of the logarithmic function we have $b^{\log_b x}=x$, hence

$$\log_a\left(b^{\log_b x}\right)=\log_a x,$$

which implies

$$\log_b x\log_a b=\log_a x,$$

and the result follows. ■

THEOREM: We have

$$\frac{d}{dx}[\log_b x]=\frac{1}{x\ln b}$$

PROOF: We have:

$$\begin{aligned}\frac{d}{dx}[\log_b x]=\lim_{h\to 0}\frac{\log_b(x+h)-\log_b x}{h}=\lim_{h\to 0}\left[\frac{1}{h}\log_b\left(\frac{x+h}{x}\right)\right] &= \lim_{h\to 0}\left[\frac{1}{h}\log_b\left(1+\frac{h}{x}\right)\right]\\ &= \lim_{h\to 0}\left[\frac{1}{x}\frac{x}{h}\log_b\left(1+\frac{h}{x}\right)\right]\\ &= \frac{1}{x}\lim_{h\to 0}\left[\log_b\left(1+\frac{h}{x}\right)^{x/h}\right]\\ &= \frac{1}{x}\lim_{h\to 0}\left[\log_b\left(1+\frac{h}{x}\right)^{1/(h/x)}\right]\\ &= \frac{1}{x}\log_b e\\ &= \frac{1}{x}\frac{\ln e}{\ln b}\\ &= \frac{1}{x\ln b}\end{aligned}$$

■

COROLLARY: We have

$$\frac{d}{dx}[\ln x]=\frac{1}{x}$$

REMARK: In general,

$$(\log_b u)' = \frac{1}{u \ln b} \cdot u' \quad \text{and} \quad (\ln u)' = \frac{1}{u} \cdot u',$$

EXAMPLE:

(a) $[\ln(x^2+1)]' = \frac{1}{x^2+1} \cdot (x^2+1)' = \frac{2x}{x^2+1}$

(b) $[\ln(\ln x)]' = \frac{1}{\ln x} \cdot (\ln x)' = \frac{1}{x \ln x}$

(c) $\left[\ln\left(\frac{x}{1+x^2}\right)\right]' = \frac{1+x^2}{x} \cdot \left(\frac{x}{1+x^2}\right)' = \frac{1+x^2}{x} \cdot \frac{x'(1+x^2) - x(1+x^2)'}{(1+x^2)^2} = \frac{1+x^2}{x} \cdot \frac{1-x^2}{(1+x^2)^2}$

EXAMPLE: Find the derivative of

$$y = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}$$

SOLUTION: We have

$$\ln y = \ln\left(\frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\right) = 2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2),$$

therefore

$$(\ln y)' = \left(2\ln x + \frac{1}{3}\ln(7x-14) - 4\ln(1+x^2)\right)',$$

so

$$\frac{1}{y} \cdot y' = \frac{2}{x} + \frac{7}{3(7x-14)} - 4\frac{2x}{1+x^2} = \frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}.$$

It follows that

$$y' = y\left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right) = \frac{x^2 \sqrt[3]{7x-14}}{(1+x^2)^4}\left(\frac{2}{x} + \frac{1}{3x-6} - \frac{8x}{1+x^2}\right).$$

EXAMPLE: Find the derivative of

$$y = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}$$

SOLUTION: We have

$$\ln y = \ln\left(\frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\right) = \frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \ln(x^6-7x+5),$$

therefore

$$(\ln y)' = \left(\frac{1}{3}\ln(x^2-8) + \frac{1}{2}\ln(x^3+1) - \ln(x^6-7x+5)\right)',$$

so

$$\frac{1}{y} \cdot y' = \frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}.$$

It follows that

$$y' = y\left(\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}\right) = \frac{(x^2-8)^{1/3}\sqrt{x^3+1}}{x^6-7x+5}\left(\frac{2x}{3(x^2-8)} + \frac{3x^2}{2(x^3+1)} - \frac{6x^5-7}{x^6-7x+5}\right)$$

THEOREM: We have

$$\frac{d}{dx}[b^x] = b^x \ln b$$

In particularly, if $b = e$, then

$$(e^x)' = e^x.$$

PROOF: Let $y = b^x$. We should prove that $y' = b^x \ln b$. We have:

$$y = b^x \quad \Longrightarrow \quad \ln y = x \ln b,$$

therefore

$$(\ln y)' = (x \ln b)' \quad \Longrightarrow \quad \frac{1}{y} \cdot y' = \ln b,$$

so

$$y' = y \ln b = b^x \ln b. \ \blacksquare$$

REMARK: In general,

$$(b^u)' = b^u \ln b \cdot u' \quad \text{and} \quad (e^u)' = e^u \cdot u'.$$

EXAMPLE: $(2^{\sin 2x})' = 2^{\sin 2x} \ln 2 \cdot (\sin 2x)' = 2^{\sin 2x} \ln 2 \cdot \cos 2x \cdot 2$