Section 4.1 - Inverse Functions

DEFINITION: If the functions f and g satisfy the two conditions

$$\begin{aligned} g(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(g(y)) &= y \quad \text{for every } y \text{ in the domain of } g \end{aligned} \qquad (*)$$

then we say that f and g are **inverses**. Moreover, we call $\boldsymbol{f}$ **an inverse function for** $\boldsymbol{g}$ and $\boldsymbol{g}$ **an inverse function for** $\boldsymbol{f}$**.**

NOTATION: The inverse of a function f is commonly denoted by f^{-1}.

So, we can reformulate $(*)$ as

$$\begin{aligned} f^{-1}(f(x)) &= x \quad \text{for every } x \text{ in the domain of } f \\ f(f^{-1}(x)) &= x \quad \text{for every } x \text{ in the domain of } f^{-1} \end{aligned}$$

EXAMPLE:

1. Let $f(x) = x^3$, then $f^{-1}(x) = \sqrt[3]{x}$, since
$$f^{-1}(f(x)) = \sqrt[3]{x^3} = x \quad \text{and} \quad f(f^{-1}(x)) = (\sqrt[3]{x})^3 = x.$$
2. Let $f(x) = x^3 + 1$, then $f^{-1}(x) = \sqrt[3]{x-1}$, since
$$f^{-1}(f(x)) = \sqrt[3]{(x^3+1)-1} = x \quad \text{and} \quad f(f^{-1}(x)) = (\sqrt[3]{x-1})^3 + 1 = x.$$
3. Let $f(x) = 2x$, then $f^{-1}(x) = \frac{1}{2}x$, since
$$f^{-1}(f(x)) = \frac{1}{2}(2x) = x \quad \text{and} \quad f(f^{-1}(x)) = 2\left(\frac{1}{2}x\right) = x.$$
4. Let $f(x) = x$, then $f^{-1}(x) = x$, since
$$f^{-1}(f(x)) = x \quad \text{and} \quad f(f^{-1}(x)) = x.$$
5. Let $f(x) = 7x + 2$, then $f^{-1}(x) = \frac{x-2}{7}$, since
$$f^{-1}(f(x)) = \frac{7x+2-2}{7} = x \quad \text{and} \quad f(f^{-1}(x)) = 7\frac{x-2}{7} + 2 = x.$$
6. Let $f(x) = 9x + 4$, then $f^{-1}(x) =$

7. Let $f(x) = \sqrt[3]{3x-2}$, then $f^{-1}(x) =$

8. Let $f(x) = (3x-2)^5 + 2$, then $f^{-1}(x) =$

9. Let $f(x) = \sqrt{x}$, then $f^{-1}(x) =$

IMPORTANT:

$$\text{domain of } f^{-1} = \text{range of } f$$
$$\text{range of } f^{-1} = \text{domain of } f$$

EXAMPLE:

1. Let $f(x) = \sqrt{x}$, then $f^{-1}(x) = x^2, \ x \geq 0$.
2. Let $f(x) = \sqrt{3x-2}$, then $f^{-1}(x) = \dfrac{1}{3}(x^2+2), \ x \geq 0$.
3. Let $f(x) = \sqrt[4]{x-1}$, then $f^{-1}(x) = x^4+1, \ x \geq 0$.
4. Let $f(x) = \sqrt{x+5}+1$, then $f^{-1}(x) = (x-1)^2-5, \ x \geq 1$.
5. Let $f(x) = \sqrt[4]{2x-7}+5$, then $f^{-1}(x) =$

6. Let $f(x) = (x+1)^2$, then $f^{-1}(x) =$

THEOREM **(The Horizontal Line Test):** A function f has an inverse function if and only if its graph is cut at most once by any horizontal line.

EXAMPLE:

1. The functions $f(x) = x^2, \ f(x) = (x+1)^2, \ f(x) = (2x-3)^4$ are not invertible.
2. Let $f(x) = x^2, x \geq 0$. Then $f^{-1}(x) = \sqrt{x}, x \geq 0$.
3. Let $f(x) = x^2, x \geq 2$. Then $f^{-1}(x) = \sqrt{x}, x \geq 4$.
4. Let $f(x) = x^2, x < -3$. Then $f^{-1}(x) = \sqrt{x}, x > 9$.
5. The function $f(x) = x^2, x > -1$ is not invertible.
6. Let $f(x) = (x+1)^2, x > 3$. Then $f^{-1}(x) =$

7. Let $f(x) = (1+2x)^2, x \leq -1$. Then $f^{-1}(x) =$

THEOREM: If f has an inverse function f^{-1}, then the graphs of $y = f(x)$ and $y = f^{-1}(x)$ are reflections of one another about the line $y = x$; that is, each is the mirror image of the other with respect to that line.

THEOREM: If the domain of a function f is an interval on which $f'(x) > 0$ or on which $f'(x) < 0$, then f has an inverse function.

EXAMPLE:

1. The function $f(x) = x^5 + x + 1$ is invertible, since $f'(x) = 5x^4 + 1 > 1$.
2. The function $f(x) = -x^9 - x^5 - x$ is...

THEOREM: Suppose that f is a function with domain D and range R. If D is an interval and f is continuous and one-to-one on D, then R is an interval and the inverse of f is continuous on R.

THEOREM **(Differentiability of Inverse Functions):** Suppose that f is a function whose domain D is an open interval, and let R be the range of f. If f is differentiable and one-to-one on D, then f^{-1} is differentiable at any value x in R for which $f'(f^{-1}(x)) \neq 0$. Furthermore, if x is in R with $f'(f^{-1}(x)) \neq 0$, then

$$\frac{d}{dx}[f^{-1}(x)] = \frac{1}{f'(f^{-1}(x))} \qquad (**)$$

COROLLARY: If the domain of a function f is an interval on which $f'(x) > 0$ or on which $f'(x) < 0$, then f has an inverse function f^{-1} and $f^{-1}(x)$ is differentiable at any value x in the range of f. The derivative of f^{-1} is given by formula $(**)$.