Section 3.3/3.4 - Techniques Of Differentiation/Derivatives Of Trigonometric Functions

THEOREM: Let c be any real number and n be any integer. If f and g are differentiable at x, then so are $f+g,\ f-g,\ f\cdot g$ and

1. $(cf)' = cf'$
2. $(f \pm g)' = f' \pm g'$
3. $(fg)' = f'g + fg'$
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, provided $g(x) \neq 0$.
5. $c' = 0$
6. $(x^n)' = nx^{n-1}$
7. $(\sin x)' = \cos x$
8. $(\cos x)' = -\sin x$
9. $(\tan x)' = \sec^2 x$
10. $(\cot x)' = -\csc^2 x$
11. $(\sec x)' = \sec x \tan x$
12. $(\csc x)' = -\csc x \cot x$

EXAMPLE:

1. $(3x-5)' = (3x)' - 5' = 3x' - 5' = 3\cdot 1 - 0 = 3$

2. $(\sqrt{x})' = \left(x^{1/2}\right)' = \dfrac{1}{2}x^{1/2-1} = \dfrac{1}{2}x^{-1/2}$

3. $\left(\dfrac{1}{\sqrt{x}}\right)' = \left(\dfrac{1}{x^{1/2}}\right)' = \left(x^{-1/2}\right)' = -\dfrac{1}{2}x^{-1/2-1} = -\dfrac{1}{2}x^{-3/2}$

4. $$(-3x^{-8} + 2\sqrt{x})' = (-3x^{-8})' + (2\sqrt{x})' = -3(x^{-8})' + 2(\sqrt{x})' = -3(-8)x^{-8-1} + 2\frac{1}{2}x^{-1/2} = 24x^{-9} + x^{-1/2}$$

5. $$[(x^3+7x^2-8)(2x^{-3}+x^{-4})]' = (x^3+7x^2-8)'(2x^{-3}+x^{-4}) + (x^3+7x^2-8)(2x^{-3}+x^{-4})'$$
$$= (3x^2+14x)(2x^{-3}+x^{-4}) + (x^3+7x^2-8)(-6x^{-4}-4x^{-5})$$

6. $$\left[\left(\frac{3x+2}{x+1}\right)(x^{-5}+1)\right]' = \left(\frac{3x+2}{x+1}\right)'(x^{-5}+1) + \left(\frac{3x+2}{x+1}\right)(x^{-5}+1)'$$
$$= \frac{(3x+2)'(x+1) - (3x+2)(x+1)'}{(x+1)^2}(x^{-5}+1) + \left(\frac{3x+2}{x+1}\right)(x^{-5}+1)'$$
$$= \frac{3(x+1) - (3x+2)}{(x+1)^2}(x^{-5}+1) + \left(\frac{3x+2}{x+1}\right)(-5x^{-6})$$

7. $$(x^3\sin x - 5\cos x)' = (x^3 \sin x)' - 5(\cos x)' = (x^3)'\sin x + x^3(\sin x)' - 5(\cos x)'$$
$$= 3x^2 \sin x + x^3 \cos x + 5\sin x$$

8. $$\left(\frac{\sin x \sec x}{1 + x\tan x}\right)' = \frac{(\sin x \sec x)'(1+x\tan x) - (\sin x \sec x)(1 + x\tan x)'}{(1+x\tan x)^2}$$
$$= \frac{[(\sin x)'\sec x + \sin x(\sec x)'](1+x\tan x) - (\sin x\sec x)(x'\tan x + x(\tan x)')}{(1+x\tan x)^2}$$
$$= \frac{[\cos x \sec x + \sin x \sec x \tan x](1 + x\tan x) - (\sin x \sec x)(\tan x + x\sec^2 x)}{(1+x\tan x)^2}$$

Section 3.5 - The Chain Rule

THEOREM (The Chain Rule): If g is differentiable at x and f is differentiable at $g(x)$, then the composition $f \circ g$ is differentiable at x. Moreover,

$$(f \circ g)'(x) = f'(g(x))g'(x).$$

Alternatively, if

$$y = f(g(x)) \quad \text{and} \quad u = g(x)$$

then $y = f(u)$ and

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

THEOREM: Let c be any real number and n be any integer. If f, g and u are differentiable at x, then so are $f + g,\ f - g,\ f \cdot g$ and

1. $(cf)' = cf'$
2. $(f \pm g)' = f' \pm g'$
3. $(fg)' = f'g + fg'$
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, provided $g(x) \neq 0$.
5. $c' = 0$
6. $(u^n)' = nu^{n-1} \cdot u'$
7. $(\sin u)' = \cos u \cdot u'$
8. $(\cos u)' = -\sin u \cdot u'$
9. $(\tan u)' = \sec^2 u \cdot u'$
10. $(\cot u)' = -\csc^2 u \cdot u'$
11. $(\sec u)' = \sec u \tan u \cdot u'$
12. $(\csc u)' = -\csc u \cot u \cdot u'$

EXAMPLE:

1. $[\sin(3x)]' = \cos(3x) \cdot (3x)' = \cos(3x) \cdot 3 = 3\cos(3x)$

2. $[4\cos(x^3)]' = 4[\cos(x^3)]' = -4\sin(x^3) \cdot (x^3)' = -12x^2 \sin(x^3)$

3. $[(x^2 - x + 1)^{23}]' = 23(x^2 - x + 1)^{22} \cdot (x^2 - x + 1)' = 23(2x - 1)(x^2 - x + 1)^{22}$

4. $$[\sqrt{x^3 + \csc x}]' = [(x^3 + \csc x)^{1/2}]' = \frac{1}{2}(x^3 + \csc x)^{-1/2}(x^3 + \csc x)'$$
$$= \frac{1}{2}(x^3 + \csc x)^{-1/2}(3x^2 - \csc x \cot x)$$

5. $$\left[\frac{1}{x^3 + 2x - 3}\right]' = [(x^3 + 2x - 3)^{-1}]' = (-1)(x^3 + 2x - 3)^{-2}(x^3 + 2x - 3)'$$
$$= -(3x^2 + 2)(x^3 + 2x - 3)^{-2}$$

6. $$[\sin(\sqrt{1 + \cos x})]' = \cos(\sqrt{1 + \cos x})[\sqrt{1 + \cos x}]'$$
$$= \cos(\sqrt{1 + \cos x})\frac{1}{2}(1 + \cos x)^{-1/2}(1 + \cos x)'$$
$$= \cos(\sqrt{1 + \cos x})\frac{1}{2}(1 + \cos x)^{-1/2}(-\sin x)$$