DEFINITION: Suppose that x_0 is a number in the domain of a function f. If

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

exists, then the value of this limit is called the **derivative of f at $x = x_0$** and is denoted by $f'(x_0)$. That is,

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If the limit of the difference quotient exists, $f'(x_0)$ is the **slope of the graph of f at the point $P(x_0, f(x_0))$** (or at $x = x_0$). If this limit does not exist, then the slope of the graph of f is **undefined** at P (or at $x = x_0$).

REMARK: Sometimes instead of

$$f'(x_0) = \lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

we write

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

EXAMPLE: Find the slope of the graph of $y = x^2 + 1$ at the point $(2, 5)$, and use it to find the equation of the tangent line to $y = x^2 + 1$ at $x = 2$.

SOLUTION: The equation of the tangent line to $y = f(x)$ at $x = x_0$ is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Since $x_0 = 2$ and $f(x_0) = 2^2 + 1 = 5$, we have $y - 5 = f'(2)(x - 2)$, so to solve the problem, we should find $f'(x_0)$. We have

$$\begin{aligned} f'(2) &= \lim_{x_1 \to 2} \frac{f(x_1) - f(2)}{x_1 - 2} \\ &= \lim_{x_1 \to 2} \frac{x_1^2 + 1 - (2^2 + 1)}{x_1 - 2} \\ &= \lim_{x_1 \to 2} \frac{x_1^2 - 4}{x_1 - 2} \\ &= \lim_{x_1 \to 2} \frac{(x_1 - 2)(x_1 + 2)}{x_1 - 2} \\ &= \lim_{x_1 \to 2} (x_1 + 2) = 4. \end{aligned}$$

So, the equation of the tangent line to $y = x^2 + 1$ at $x = 2$ is $y - 5 = 4(x - 2)$.

REMARK: We can find $f'(2)$ using $f'(x_0) = \lim_{h\to 0} \frac{f(x_0+h)-f(x_0)}{h}$. In fact,

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2+1-(2^2+1)}{h} \\ &= \lim_{h\to 0} \frac{h^2+4h}{h} \\ &= \lim_{h\to 0}(h+4) = 4. \end{aligned}$$

PROBLEM: Find the derivative of $f(x) = x^3 - x$.

SOLUTION: We have

$$\begin{aligned}
f'(x) &= \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{x_1^3 - x_1 - (x^3 - x)}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{x_1^3 - x^3 - (x_1 - x)}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{(x_1 - x)(x_1^2 + x_1 x + x^2) - (x_1 - x)}{x_1 - x} \\
&= \lim_{x_1 \to x} (x_1^2 + x_1 x + x^2 - 1) = 3x^2 - 1.
\end{aligned}$$

PROBLEM: Find the derivative of $f(x) = \sqrt{x}$.

SOLUTION: We have

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{\sqrt{x_1} - \sqrt{x}}{x_1 - x}$$

$$= \lim_{x_1 \to x} \frac{(\sqrt{x_1} - \sqrt{x})(\sqrt{x_1} + \sqrt{x})}{(x_1 - x)(\sqrt{x_1} + \sqrt{x})}$$

$$= \lim_{x_1 \to x} \frac{x_1 - x}{(x_1 - x)(\sqrt{x_1} + \sqrt{x})}$$

$$= \lim_{x_1 \to x} \frac{1}{\sqrt{x_1} + \sqrt{x}} = \frac{1}{2\sqrt{x}}.$$

PROBLEM: Find the derivative of $f(x) = \dfrac{1}{x}$.

SOLUTION: We have

$$f'(x) = \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \to x} \frac{\dfrac{1}{x_1} - \dfrac{1}{x}}{x_1 - x}$$

$$= \lim_{x_1 \to x} \frac{\dfrac{x - x_1}{x_1 x}}{x_1 - x} = \lim_{x_1 \to x} \frac{x - x_1}{(x_1 - x)x_1 x}$$

$$= -\lim_{x_1 \to x} \frac{1}{x_1 x} = -\frac{1}{x^2}.$$

PROBLEM: Find the derivative of $f(x) = \dfrac{1}{\sqrt{x}}$.

SOLUTION: We have

$$
\begin{aligned}
f'(x) &= \lim_{x_1 \to x} \frac{f(x_1) - f(x)}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{\dfrac{1}{\sqrt{x_1}} - \dfrac{1}{\sqrt{x}}}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{\dfrac{\sqrt{x} - \sqrt{x_1}}{\sqrt{x_1}\sqrt{x}}}{x_1 - x} \\
&= \lim_{x_1 \to x} \frac{\sqrt{x} - \sqrt{x_1}}{(x_1 - x)\sqrt{x_1}\sqrt{x}} \\
&= \lim_{x_1 \to x} \frac{(\sqrt{x} - \sqrt{x_1})(\sqrt{x} + \sqrt{x_1})}{(x_1 - x)\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} \\
&= \lim_{x_1 \to x} \frac{x - x_1}{(x_1 - x)\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} \\
&= -\lim_{x_1 \to x} \frac{1}{\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} \\
&= -\frac{1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} \\
&= -\frac{1}{2x\sqrt{x}}
\end{aligned}
$$

PROBLEM: Find the derivative of $f(x) = \sin x$.
SOLUTION: We have

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin x}{h} \\
&= [\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta] \\
&= \lim_{h\to 0} \frac{\sin x\cos h + \cos x\sin h - \sin x}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1) + \cos x\sin h}{h} \\
&= \lim_{h\to 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h\to 0} \frac{\cos x\sin h}{h} \\
&= \sin x \lim_{h\to 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 = \cos x
\end{aligned}
$$

PROBLEM: Find the derivative of $f(x) = |x|$ at $x = -2$ and $x = 10$. Prove that f is not differentiable at $x = 0$.

SOLUTION: We have

$$f'(-2) = \lim_{x_1 \to -2} \frac{f(x_1) - f(-2)}{x_1 - (-2)}$$

$$= \lim_{x_1 \to -2} \frac{|x_1| - |-2|}{x_1 + 2}$$

$$= \lim_{x_1 \to -2} \frac{-x_1 - 2}{x_1 + 2}$$

$$= -\lim_{x_1 \to -2} \frac{x_1 + 2}{x_1 + 2}$$

$$= -1$$

Similarly,

$$\begin{aligned}
f'(10) &= \lim_{x_1 \to 10} \frac{f(x_1) - f(10)}{x_1 - 10} \\
&= \lim_{x_1 \to 10} \frac{|x_1| - |10|}{x_1 - 10} \\
&= \lim_{x_1 \to 10} \frac{x_1 - 10}{x_1 - 10} \\
&= 1
\end{aligned}$$

We now prove that f is not differentiable at $x = 0$. We have

$$f'(0) = \lim_{x_1 \to 0} \frac{f(x_1) - f(0)}{x_1 - 0} = \lim_{x_1 \to 0} \frac{|x_1| - |0|}{x_1 - 0}$$

$$= \lim_{x_1 \to 0} \frac{|x_1|}{x_1}.$$

This limit does not exist, since

$$\lim_{x_1 \to 0^-} \frac{|x_1|}{x_1} = \lim_{x_1 \to 0^-} \frac{-x_1}{x_1} = -1$$

and

$$\lim_{x_1 \to 0^+} \frac{|x_1|}{x_1} = \lim_{x_1 \to 0^+} \frac{x_1}{x_1} = 1$$

Section 3.3 - Techniques Of Differentiation

THEOREM: Let c be any real number and n be any integer. If f and g are differentiable at x, then so are $f+g,\ f-g,\ f\cdot g$ and

1. $(cf)' = cf'$
2. $(f \pm g)' = f' \pm g'$
3. $(fg)' = f'g'$
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'}{g'}$, provided $g(x) \neq 0$
5. $c' = 0$
6. $(x^n)' = nx^{n-1}$
7. $(\sin x)' = \cos x$
8. $(\cos x)' = \sin x$
9. $(\tan x)' = \sec^2 x$
10. $(\cot x)' = -\csc^2 x$
11. $(\sec x)' = \sec x \tan x$
12. $(\csc x)' = -\csc x \cot x$

THEOREM: Let c be any real number and n be any integer. If f and g are differentiable at x, then so are $f+g,\ f-g,\ f\cdot g$ and

1. $(cf)' = cf'$
2. $(f \pm g)' = f' \pm g'$
3. $(fg)' = f'g + fg'$
4. $\left(\dfrac{f}{g}\right)' = \dfrac{f'g - fg'}{g^2}$, provided $g(x) \neq 0$
5. $c' = 0$
6. $(x^n)' = nx^{n-1}$
7. $(\sin x)' = \cos x$
8. $(\cos x)' = -\sin x$
9. $(\tan x)' = \sec^2 x$
10. $(\cot x)' = -\csc^2 x$
11. $(\sec x)' = \sec x \tan x$
12. $(\csc x)' = -\csc x \cot x$