

Section 3.2 - The Derivative

DEFINITION: Suppose that x_0 is a number in the domain of a function f . If

$$\lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

exists, then the value of this limit is called the **derivative of f at $x = x_0$** and is denoted by $f'(x_0)$. That is,

$$f'(x_0) = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If the limit of the difference quotient exists, $f'(x_0)$ is the **slope of the graph of f at the point $P(x_0, f(x_0))$** (or at $x = x_0$). If this limit does not exist, then the slope of the graph of f is **undefined** at P (or at $x = x_0$).

REMARK: Sometimes instead of $f'(x_0) = \lim_{x_1 \rightarrow x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ we write $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$.

EXAMPLE: Find the slope of the graph of $y = x^2 + 1$ at the point $(2, 5)$, and use it to find the equation of the tangent line to $y = x^2 + 1$ at $x = 2$.

SOLUTION: The equation of the tangent line to $y = f(x)$ at $x = x_0$ is

$$y - f(x_0) = f'(x_0)(x - x_0).$$

Since $x_0 = 2$ and $f(x_0) = 2^2 + 1 = 5$, we have $y - 5 = f'(2)(x - 2)$, so to solve the problem, we should find $f'(x_0)$. We have

$$f'(2) = \lim_{x_1 \rightarrow 2} \frac{f(x_1) - f(2)}{x_1 - 2} = \lim_{x_1 \rightarrow 2} \frac{x_1^2 + 1 - (2^2 + 1)}{x_1 - 2} = \lim_{x_1 \rightarrow 2} \frac{x_1^2 - 4}{x_1 - 2} = \lim_{x_1 \rightarrow 2} \frac{(x_1 - 2)(x_1 + 2)}{x_1 - 2} = \lim_{x_1 \rightarrow 2} (x_1 + 2) = 4.$$

So, the equation of the tangent line to $y = x^2 + 1$ at $x = 2$ is $y - 5 = 4(x - 2)$.

REMARK: We can find $f'(2)$ using $f'(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$. In fact,

$$f'(2) = \lim_{h \rightarrow 0} \frac{f(2 + h) - f(2)}{h} = \lim_{h \rightarrow 0} \frac{(2 + h)^2 + 1 - (2^2 + 1)}{h} = \lim_{h \rightarrow 0} \frac{h^2 + 4h}{h} = \lim_{h \rightarrow 0} (h + 4) = 4.$$

PROBLEM: Find the derivative of $f(x) = x^3 - x$.

SOLUTION: We have

$$\begin{aligned} f'(x) &= \lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{x_1^3 - x_1 - (x^3 - x)}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{x_1^3 - x^3 - (x_1 - x)}{x_1 - x} \\ &= \lim_{x_1 \rightarrow x} \frac{(x_1 - x)(x_1^2 + x_1x + x^2) - (x_1 - x)}{x_1 - x} = \lim_{x_1 \rightarrow x} (x_1^2 + x_1x + x^2 - 1) = 3x^2 - 1. \end{aligned}$$

PROBLEM: Find the derivative of $f(x) = \sqrt{x}$.

SOLUTION: We have

$$\begin{aligned} f'(x) &= \lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\sqrt{x_1} - \sqrt{x}}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{(\sqrt{x_1} - \sqrt{x})(\sqrt{x_1} + \sqrt{x})}{(x_1 - x)(\sqrt{x_1} + \sqrt{x})} \\ &= \lim_{x_1 \rightarrow x} \frac{x_1 - x}{(x_1 - x)(\sqrt{x_1} + \sqrt{x})} = \lim_{x_1 \rightarrow x} \frac{1}{\sqrt{x_1} + \sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

PROBLEM: Find the derivative of $f(x) = \frac{1}{x}$.

SOLUTION: We have

$$f'(x) = \lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\frac{1}{x_1} - \frac{1}{x}}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\frac{x - x_1}{x_1x}}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{x - x_1}{(x_1 - x)x_1x} = - \lim_{x_1 \rightarrow x} \frac{1}{x_1x} = -\frac{1}{x^2}.$$

PROBLEM: Find the derivative of $f(x) = \frac{1}{\sqrt{x}}$.

SOLUTION: We have

$$\begin{aligned} f'(x) &= \lim_{x_1 \rightarrow x} \frac{f(x_1) - f(x)}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\frac{1}{\sqrt{x_1}} - \frac{1}{\sqrt{x}}}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\frac{\sqrt{x} - \sqrt{x_1}}{\sqrt{x_1}\sqrt{x}}}{x_1 - x} = \lim_{x_1 \rightarrow x} \frac{\sqrt{x} - \sqrt{x_1}}{(x_1 - x)\sqrt{x_1}\sqrt{x}} \\ &= \lim_{x_1 \rightarrow x} \frac{(\sqrt{x} - \sqrt{x_1})(\sqrt{x} + \sqrt{x_1})}{(x_1 - x)\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} = \lim_{x_1 \rightarrow x} \frac{x - x_1}{(x_1 - x)\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} \\ &= - \lim_{x_1 \rightarrow x} \frac{1}{\sqrt{x_1}\sqrt{x}(\sqrt{x} + \sqrt{x_1})} = - \frac{1}{\sqrt{x}\sqrt{x}(\sqrt{x} + \sqrt{x})} = - \frac{1}{2x\sqrt{x}} \end{aligned}$$

PROBLEM: Find the derivative of $f(x) = \sin x$.

SOLUTION: We have

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\sin(x+h) - \sin x}{h} = [\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta] \\ &= \lim_{h \rightarrow 0} \frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h} \\ &= \lim_{h \rightarrow 0} \frac{\sin x(\cos h - 1)}{h} + \lim_{h \rightarrow 0} \frac{\cos x \sin h}{h} \\ &= \sin x \lim_{h \rightarrow 0} \frac{(\cos h - 1)}{h} + \cos x \lim_{h \rightarrow 0} \frac{\sin h}{h} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x \end{aligned}$$

PROBLEM: Find the derivative of $f(x) = |x|$ at $x = -2$ and $x = 10$. Prove that f is not differentiable at $x = 0$.

SOLUTION: We have

$$f'(-2) = \lim_{x_1 \rightarrow -2} \frac{f(x_1) - f(-2)}{x_1 - (-2)} = \lim_{x_1 \rightarrow -2} \frac{|x_1| - |-2|}{x_1 + 2} = \lim_{x_1 \rightarrow -2} \frac{-x_1 - 2}{x_1 + 2} = - \lim_{x_1 \rightarrow -2} \frac{x_1 + 2}{x_1 + 2} = -1$$

Similarly,

$$f'(10) = \lim_{x_1 \rightarrow 10} \frac{f(x_1) - f(10)}{x_1 - 10} = \lim_{x_1 \rightarrow 10} \frac{|x_1| - |10|}{x_1 - 10} = \lim_{x_1 \rightarrow 10} \frac{x_1 - 10}{x_1 - 10} = 1$$

We now prove that f is not differentiable at $x = 0$. We have

$$f'(0) = \lim_{x_1 \rightarrow 0} \frac{f(x_1) - f(0)}{x_1 - 0} = \lim_{x_1 \rightarrow 0} \frac{|x_1| - |0|}{x_1 - 0} = \lim_{x_1 \rightarrow 0} \frac{|x_1|}{x_1}.$$

This limit does not exist, since

$$\lim_{x_1 \rightarrow 0^-} \frac{|x_1|}{x_1} = \lim_{x_1 \rightarrow 0^-} \frac{-x_1}{x_1} = -1 \quad \text{and} \quad \lim_{x_1 \rightarrow 0^+} \frac{|x_1|}{x_1} = \lim_{x_1 \rightarrow 0^+} \frac{x_1}{x_1} = 1$$

Section 3.3 - Techniques Of Differentiation

THEOREM: Let c be any real number and n be any integer. If f and g are differentiable at x , then so are $f + g$, $f - g$, $f \cdot g$ and

- | | | |
|---|--------------------------|----------------------------------|
| 1. $(cf)' = cf'$ | 5. $c' = 0$ | 9. $(\tan x)' = \sec^2 x$ |
| 2. $(f \pm g)' = f' \pm g'$ | 6. $(x^n)' = nx^{n-1}$ | 10. $(\cot x)' = -\csc^2 x$ |
| 3. $(fg)' = f'g'$ | 7. $(\sin x)' = \cos x$ | 11. $(\sec x)' = \sec x \tan x$ |
| 4. $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$, provided $g(x) \neq 0$. | 8. $(\cos x)' = -\sin x$ | 12. $(\csc x)' = -\csc x \cot x$ |