Section 2.4 - Limits (Discussed More Rigorously)

LIMITS (AN INFORMAL VIEW): If the values of $f(x)$ can be made as close as we like to L by taking values of x sufficiently close to a (but not equal to a), then we write

$$\lim_{x\to a} f(x) = L$$

which is read "the limit of $f(x)$ as x approaches a is L."

DEFINITION: Let $f(x)$ be defined for all x in some open interval containing the number a, with the possible exception that $f(x)$ need not to be defined at a. We will write

$$\lim_{x\to a} f(x) = L$$

if given any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|f(x) - L| < \epsilon \quad \text{if} \quad 0 < |x - a| < \delta$$

EXAMPLE: Prove that $\lim_{x\to 2}(3x - 5) = 1$.

PROOF: We need to prove that for any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|3x - 5 - 1| < \epsilon \quad \text{if} \quad 0 < |x - 2| < \delta$$

or

$$|3x - 6| < \epsilon \quad \text{if} \quad 0 < |x - 2| < \delta$$

or

$$3|x - 2| < \epsilon \quad \text{if} \quad 0 < |x - 2| < \delta$$

or

$$|x - 2| < \epsilon/3 \quad \text{if} \quad 0 < |x - 2| < \delta$$

This is true if $\delta = \epsilon/3$.

EXAMPLE: Prove that $\lim_{x\to 0} \sqrt[3]{x} = 0$.

PROOF: We need to prove that for any number $\epsilon > 0$ we can find a number $\delta > 0$ such that

$$|\sqrt[3]{x} - 0| < \epsilon \quad \text{if} \quad 0 < |x - 0| < \delta$$

or

$$|\sqrt[3]{x}| < \epsilon \quad \text{if} \quad 0 < |x| < \delta$$

or

$$|x| < \epsilon^3 \quad \text{if} \quad 0 < |x| < \delta$$

This is true if $\delta = \epsilon^3$.

LIMITS AT INFINITY (AN INFORMAL VIEW): If the the values of $f(x)$ eventually get closer and closer to a number L as x increases without bound, then we write

$$\lim_{x\to+\infty} f(x)=L.$$

Similarly, if the the values of $f(x)$ eventually get closer and closer to a number L as x decreases without bound, then we write

$$\lim_{x\to-\infty} f(x)=L.$$

DEFINITION: Let $f(x)$ be defined for all x in some infinite open interval extending in the positive x-direction. We will write

$$\lim_{x\to+\infty} f(x)=L$$

if given any number $\epsilon>0$, there corresponds a positive number N such that

$$|f(x)-L|<\epsilon \quad \text{if} \quad x>N$$

DEFINITION: Let $f(x)$ be defined for all x in some infinite open interval extending in the negative x-direction. We will write

$$\lim_{x\to-\infty} f(x)=L$$

if given any number $\epsilon>0$, there corresponds a negative number N such that

$$|f(x)-L|<\epsilon \quad \text{if} \quad x<N$$

EXAMPLE: Prove that $\lim\limits_{x\to+\infty} \dfrac{1}{x}=0.$

PROOF: We need to prove that for any number $\epsilon>0$ we can find a number $N>0$ such that

$$\left|\frac{1}{x}-0\right|<\epsilon \quad \text{if} \quad x>N$$

or

$$\frac{1}{x}<\epsilon \quad \text{if} \quad x>N$$

or

$$x>\frac{1}{\epsilon} \quad \text{if} \quad x>N$$

This is true if $N=\dfrac{1}{\epsilon}$.