On approximation of real, complex, and $p$-adic numbers by algebraic numbers of bounded degree

BY

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I. On approximation by rational numbers

**Theorem 1** (Dirichlet, 1842). For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2}.$$
Example: Let $\xi = e$. Consider the continued fraction expansion:

$$e = 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1 + \frac{1}{1 + \ldots}}}}$$

We have

$$2 + \frac{1}{1} = 3 \quad 2 + \frac{1}{1 + \frac{1}{2}} = \frac{8}{3} \quad 2 + \frac{1}{1 + \frac{1}{2 + \frac{1}{1}}} = \frac{11}{4}$$

and so on.
The first convergents are:

\[
\begin{align*}
\delta_1 &= 3 & |e - 3| &< 1 \\
\delta_2 &= \frac{8}{3} & |e - \frac{8}{3}| &< \frac{1}{3^2} \\
\delta_3 &= \frac{11}{4} & |e - \frac{11}{4}| &< \frac{1}{4^2} \\
\delta_4 &= \frac{19}{7} & |e - \frac{19}{7}| &< \frac{1}{7^2} \\
\delta_5 &= \frac{87}{32} & |e - \frac{87}{32}| &< \frac{1}{32^2} \\
\delta_6 &= \frac{106}{39} & |e - \frac{106}{39}| &< \frac{1}{39^2}
\end{align*}
\]
We also note that

\[ \delta_1 = 3 \quad |e - 3| < \frac{1}{2 \cdot 1^2} \]

\[ \delta_2 = \frac{8}{3} \quad |e - \frac{8}{3}| < \frac{1}{2 \cdot 3^2} \]

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\[ \delta_4 = \frac{19}{7} \quad |e - \frac{19}{7}| < \frac{1}{2 \cdot 7^2} \]

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\[ \delta_6 = \frac{106}{39} \quad |e - \frac{106}{39}| < \frac{1}{39^2} \]
Theorem 2. For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{2q^2}.$$
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**Theorem 1.** For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2}.$$
Finally, some convergents give even better approximation:

\[
\begin{align*}
\delta_1 &= 3 \quad \left| e - 3 \right| < \frac{1}{\sqrt{5} \cdot 1^2} \\
\delta_2 &= \frac{8}{3} \quad \left| e - \frac{8}{3} \right| < \frac{1}{2 \cdot 3^2} \\
\delta_3 &= \frac{11}{4} \quad \left| e - \frac{11}{4} \right| < \frac{1}{4^2} \\
\delta_4 &= \frac{19}{7} \quad \left| e - \frac{19}{7} \right| < \frac{1}{\sqrt{5} \cdot 7^2} \\
\delta_5 &= \frac{87}{32} \quad \left| e - \frac{87}{32} \right| < \frac{1}{2 \cdot 32^2} \\
\delta_6 &= \frac{106}{39} \quad \left| e - \frac{106}{39} \right| < \frac{1}{39^2}
\end{align*}
\]
Theorem 3 \textit{(Hurwitz)}. For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$
\left| \xi - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.
$$
Theorem 3 (Hurwitz). For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$\left| \xi - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.$$ 

This result is the best possible.
**Theorem 3** (Hurwitz). For any real irrational number \( \xi \) there exist infinitely many rational numbers \( p/q \) such that

\[
\left| \xi - \frac{p}{q} \right| < \frac{1}{\sqrt{5}q^2}.
\]

This result is the best possible.

**Example:**

\[
\xi = \frac{1 + \sqrt{5}}{2} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \ldots}}}
\]

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II. Polynomial Interpretation

**Theorem 1.** For any real irrational number \( \xi \) there exist infinitely many rational numbers \( p/q \) such that

\[
\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2}.
\]
II. Polynomial Interpretation

**Theorem 1.** For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that

$$\left|\xi - \frac{p}{q}\right| < \frac{1}{q^2} \implies |q\xi - p| < q^{-1}.$$
II. Polynomial Interpretation

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$$\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2} \implies |q\xi - p| < q^{-1}.$$

Theorem 4. For any real irrational number $\xi$ there exist infinitely many polynomials $P \in Z[x]$ of the first degree such that

$$|P(\xi)| \ll |P|^{-1},$$

where $|P|$ denotes the height of $P$. 
II. Polynomial Interpretation

**Theorem 1.** For any real irrational number $\xi$ there exist infinitely many rational numbers $p/q$ such that
\[
\left| \xi - \frac{p}{q} \right| < \frac{1}{q^2} \implies |q\xi - p| < q^{-1}.
\]

**Theorem 5.** For any real number $\xi \notin A_n$ there exist infinitely many polynomials $P \in \mathbb{Z}[x]$ of degree $\leq n$ such that
\[
|P(\xi)| \ll |P|^{-n},
\]
where $A_n$ is the set of real algebraic numbers of degree $\leq n$. 
\[ \left| \xi - \frac{p}{q} \right| < q^{-2} \]
\[ |\xi - \frac{p}{q}| < q^{-2} \quad \longrightarrow \quad |q\xi - p| < q^{-1} \]
\[
\left| \xi - \frac{p}{q} \right| < q^{-2} \quad \rightarrow \quad \left| q\xi - p \right| < q^{-1}
\]

\[
|P(\xi)| \ll \left| P \right|^{-n}
\]
\[
\left| \xi - \frac{p}{q} \right| < q^{-2} \quad \quad \quad \quad \quad \quad \left| \xi \right| - \alpha \mid < q^{-1} \ll |P|^{-n}
\]
\[ |\xi - \frac{p}{q}| < q^{-2} \quad \rightarrow \quad |q\xi - p| < q^{-1} \]

\[ |\xi - \alpha| \ll ? \quad \leftarrow \quad |P(\xi)| \ll |P|^{-n} \]
### III. Conjecture of Wirsing

**Conjecture** (Wirsing, 1961). For any real number $\xi \notin A_n$, there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n-1+\epsilon}, \quad \epsilon > 0,$$

where $H(\alpha)$ is the height of $\alpha$.

Further W. M. Schmidt conjectured that the exponent

$$-n - 1 + \epsilon$$

can be replaced even by

$$-n - 1.$$
\[ \left| \xi - \frac{p}{q} \right| < q^{-2} \quad \rightarrow \quad \left| q\xi - p \right| < q^{-1} \]

\[ \left| \xi - \alpha \right| \ll H(\alpha)^{-n-1} \quad \leftarrow \quad \left| P(\xi) \right| \ll \left| P \right|^{-n} \]
At the moment this Conjecture is proved only for $n = 1 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-2}$ (Dirichlet)
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\[ n = 1 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-n-1} \]
\[ n = 2 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-n-1} \]

(Dirichlet)

(Davenport – Schmidt)
\[
\begin{align*}
|\xi - \frac{p}{q}| < q^{-2} & \quad \quad \rightarrow \quad \quad |q\xi - p| < q^{-1} \\
|\xi - \alpha| \ll H(\alpha)^{-n-1} & \quad \quad \quad \rightarrow \quad \quad |P(\xi)| \ll |P|^{-n}
\end{align*}
\]

At the moment this Conjecture is proved only for

\(n = 1 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-2}\) (Dirichlet)

\(n = 2 \Rightarrow |\xi - \alpha| \ll H(\alpha)^{-3}\) (Davenport – Schmidt)

\(n > 2 \Rightarrow ???\)
Consider the polynomial

\[ P(x) = a_n x^n + \ldots + a_1 x + a_0 \]

\[ = a_n (x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n). \]

Without loss of generality we can assume that \( \alpha_1 \) is the root of \( P(x) \) closest to \( \xi \).
It is known that

\[ |\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}. \]

By Theorem 3 there are infinitely many polynomials \( P \in \mathbb{Z}[x] \) of degree \( \leq n \) such that

\[ |P(\xi)| \ll |P|^{-n}. \]

Let \( n = 1 \). Then

\[ |P'(\xi)| = |a_1| \asymp |P| \quad \Rightarrow \quad |\xi - \alpha_1| \ll \frac{P^{-1}}{|P|} = |P|^{-2} \]
It is known that
\[
|\xi - \alpha_1| \ll \frac{|P(\xi)|}{|P'(\xi)|}.
\]

By Theorem 3 there are infinitely many polynomials \( P \in \mathbb{Z}[x] \) of degree \( \leq n \) such that
\[
|P(\xi)| \ll |P|^{-n}.
\]

Let \( n = 2 \). Then for some \( \delta \leq 1 \) we have
\[
|P'(\xi)| = |2a_2\xi + a_1| \asymp |P|^\delta \Rightarrow |\xi - \alpha_1| \ll \frac{|P|^{-2}}{|P|^{\delta}} = |P|^{-2-\delta}
\]

**Question**: Can one prove that the following is impossible: All polynomials with $|P(\xi)| \ll |P|^{-n}$ have a “small” derivative $|P'(\xi)| \asymp |P|^{\delta}$. 
**Question**: Can one prove that the following is impossible: All polynomials with $|P(\xi)| \ll |P|^{-n}$ have a “small” derivative $|P'(\xi)| \asymp |P|^\delta$.

**Answer**: Yes, for $n=1$ (Dirichlet, 1842) \\
n=2 (Davenport – Schmidt, 1967)
IV. Theorem of Wirsing

**Theorem 6 (Wirsing, 1961).** For any real number \( \xi \not\in A_n \) there exist infinitely many algebraic numbers \( \alpha \in A_n \) with

\[
|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}} - \lambda_n, \quad \lim_{n \to \infty} \lambda_n = 2.
\]
By Dirichlet’s Box principle there are infinitely many polynomials

\[ P(x) = a_n(x - \alpha_1) \cdot \ldots \cdot (x - \alpha_n) \]

such that \(|P(\xi)| \ll |P|^{-n}\), therefore

\[ |\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll |P|^{-n}a_n^{-1}. \]

Even if \(a_n = |P|\), we can only prove that

\[ |\xi - \alpha_1| \cdot \ldots \cdot |\xi - \alpha_n| \ll |P|^{-n-1} \ll H(\alpha_1)^{-n-1}, \]

\[ \Downarrow \]

\[ |\xi - \alpha_1| \ll H(\alpha_1)^{-n+1}?? \]

It is also clear, that the worth case for us is when

\[ |\xi - \alpha_1| = \ldots = |\xi - \alpha_n|. \]
QUESTION: Can one prove that for infinitely many polynomials $P \in \mathbb{Z}[x]$ with $|P(\xi)| \ll |P|^{-n}$, the situation

$$|\xi - \alpha_1| = \ldots = |\xi - \alpha_n|$$

is impossible?
**Question:** Can one prove that for infinitely many polynomials $P \in \mathbb{Z}[x]$ with $|P(\xi)| \ll |P|^{-n}$ the situation
\[ |\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \]
is impossible?

**Answer:** For infinitely many polynomials $P \in \mathbb{Z}[x]$ with $|P(\xi)| \ll |P|^{-n}$ we have:
\[ |\xi - \alpha_1| \ll |\xi - \alpha_2| \ll 1, \]
\[ |\xi - \alpha_3|, \ldots, |\xi - \alpha_n| \text{ are “big”}. \]
Step 1: Construct infinitely many $P, Q \in \mathbb{Z}[x]$, \(\deg P, Q \leq n\), such that

\[
\begin{align*}
|P(\xi)| &\ll |P|^{-n} \\
|Q(\xi)| &\ll |Q|^{-n} \\
|P| &\ll |Q|
\end{align*}
\]

and $P, Q$ have no common root.
Step 2. Consider the resultant of $P, Q$:

$$R(P, Q) = a_m^\ell b_\ell^m \prod_{1 \leq i, j \leq n} (\alpha_i - \beta_j).$$
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On the one hand,

$$R(P, Q) \neq 0,$$

since $P, Q$ have no common root.
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since $P, Q$ have no common root. Moreover,

$$R(P, Q) \in \mathbb{Z},$$

since $P, Q$ have integer coefficients. Therefore we get

$$|R(P, Q)| \geq 1.$$
Step 3. On the other hand,

\[ |R(P, Q)| = a_m^\ell b_\ell^m \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j|. \]
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$$|R(P, Q)| = a_m^\ell b_m^m \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j|$$

$$\ll |P|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j|.$$
Step 3. On the other hand,

\[ |R(P, Q)| = a_m^\ell b_m^m \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} \max(|\xi - \alpha_i|, |\xi - \beta_j|). \]
Step 3. On the other hand,

\[ |R(P, Q)| = \alpha_m^\ell b_m^m \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \max_{1 \leq i, j \leq n} (|\xi - \alpha_i|, |\xi - \beta_j|). \]

If

\[ |\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll |P|^{-\frac{1}{n}}, \]

\[ |\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll |P|^{-\frac{1}{n}} \]
Step 3. On the other hand,

\[ |R(P, Q)| = \alpha_m^\ell b^m \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]

\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} \max(|\xi - \alpha_i|, |\xi - \beta_j|). \]

If

\[ |\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll |P|^{-1 - \frac{1}{n}}, \]

\[ |\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll |P|^{-1 - \frac{1}{n}}, \]

then

\[ |R(P, Q)| \ll |P|^{2n} |P|^{(-1 - \frac{1}{n})n^2} = |P|^{n-n^2} < 1 \]
Step 3. On the other hand,
\[ |R(P, Q)| = a^m b^n \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]
\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} |\alpha_i - \beta_j| \]
\[ \ll |P|^{2n} \prod_{1 \leq i, j \leq n} \max(|\xi - \alpha_i|, |\xi - \beta_j|). \]

If
\[ |\xi - \alpha_1| = \ldots = |\xi - \alpha_n| \ll |P|^{-1 - \frac{1}{n}}, \]
\[ |\xi - \beta_1| = \ldots = |\xi - \beta_n| \ll |P|^{-1 - \frac{1}{n}}, \]
then
\[ |R(P, Q)| \ll |P|^{2n} |P|^{(-1 - \frac{1}{n})n^2} = |P|^{n - n^2} < 1, \]
which gives a contradiction, since \(|R(P, Q)| \geq 1\) by Step 2.
Lemma (Wirsing, 1961):

\[
|\xi - \gamma| \ll \max \left\{ \left| \frac{1}{2} P(\xi) \right| \left| Q(\xi) \right| P^{-\frac{3}{2}}, \left| \frac{1}{2} Q(\xi) \right| P^{-\frac{3}{2}} \right\},
\]

where \( \gamma \) is a root of \( P \) or \( Q \) closest to \( \xi \).

Since

\[
|P(\xi)| \ll \left| P \right|^{-n}, \quad |Q(\xi)| \ll \left| Q \right|^{-n},
\]

we get

\[
|\xi - \gamma| \ll \left| \frac{1}{2} P^{\frac{n}{2}} - n + n^{-\frac{3}{2}} \right| = \left| P^{-\frac{n}{2}} - \frac{3}{2} \right|.
\]
V. “Big Derivative” Method

Theorem 7 (Bernik-Tsishchanka, 1993). For any real number $\xi \not\in A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \lambda_n}, \quad \lim_{n \to \infty} \lambda_n = 3.$$
The following table contains the values of

$$\frac{n}{2} + \lambda_n$$

corresponding to Wirsing’s Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), and the Conjecture:

<table>
<thead>
<tr>
<th>$n$</th>
<th>1961</th>
<th>1993</th>
<th>Conj.</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>3.28</td>
<td>3.5</td>
<td>4</td>
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<tr>
<td>4</td>
<td>3.82</td>
<td>4.12</td>
<td>5</td>
</tr>
<tr>
<td>5</td>
<td>4.35</td>
<td>4.71</td>
<td>6</td>
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<td>10</td>
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<td>7.47</td>
<td>11</td>
</tr>
<tr>
<td>50</td>
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<td>27.84</td>
<td>51</td>
</tr>
<tr>
<td>100</td>
<td>51.99</td>
<td>52.92</td>
<td>101</td>
</tr>
</tbody>
</table>
Fix some $H > 0$. By Dirichlet’s Box Principle there exists an integer polynomial $P$ such that

$$|a_n| \ll H, \ldots, |a_2| \ll H,$$

$$|a_1| \ll H^{1+\varepsilon}, \quad |a_0| \ll H^{1+\varepsilon},$$

$$|P(\xi)| \ll H^{-n-\varepsilon},$$

where $\varepsilon > 0$. 

Case A: Let 
\[ \max(|a_1|, |a_0|) \gg H, \]
that is 
\[ \max(|a_1|, |a_0|) = H^{1+\delta} = |P|, \quad 0 < \delta \leq \epsilon. \]
It is clear that in this case the derivative of \( P \) is “big”, that is 
\[ |P'(\xi)| \asymp H^{1+\delta}. \quad (2) \]
We have the following well-known inequality 
\[ |\xi - \alpha| \ll \frac{|P(\xi)|}{|P'(\xi)|}, \quad (3) \]
where \( \alpha \) is the root of the polynomial \( P \) closest to \( \xi \). Substituting (1) and (2) into (3), we get 
\[ |\xi - \alpha| \ll \frac{H^{-n-\epsilon}}{H^{1+\delta}} = H^{-(1+\delta)\frac{n+1+\epsilon+\delta}{1+\delta}} = |P|^{-\frac{n+1+\epsilon+\delta}{1+\delta}} \ll H(\alpha) \]
Case B: Let
\[ \max(|a_1|, |a_0|) \ll H, \]
then
\[ |\mathcal{P}| \ll H. \tag{4} \]
Using Dirichlet’s Box we construct an integer polynomial \( Q \) such that
\[ |b_n| \ll H, \ldots, |b_2| \ll H, \quad |b_1| \ll H^{1+\epsilon}, \quad |b_0| \ll H^{1+\epsilon}, \]
\[ |Q(\xi)| \ll H^{-n-\epsilon}, \tag{5} \]
If \( \max(|b_1|, |b_0|) \gg H \), then
\[ |\xi - \beta| \ll H(\beta)^{-\frac{n+1+2\epsilon}{1+\epsilon}}. \]
If \( \max(|b_1|, |b_0|) \ll H \), then
\[ |Q| \ll H. \tag{6} \]
Then we can apply Wirsing’s Lemma:
\[ |\xi - \gamma| \ll \max \left\{ |P(\xi)|^{\frac{1}{2}} |Q(\xi)| |\mathcal{P}|^{-\frac{3}{2}}, |P(\xi)| |Q(\xi)|^{\frac{1}{2}} |\mathcal{P}|^{-\frac{3}{2}}, \right\} \]
Substituting (4), (5), (6), and \( |P(\xi)| \ll H^{-n-\epsilon} \), we get:
\[ |\xi - \gamma| \ll H^{-\frac{n}{2}} - \frac{3}{2} - \frac{3}{2}\epsilon \ll H(\gamma)^{-\frac{n}{2}} - \frac{3}{2} - \frac{3}{2}\epsilon. \]
Let us compare estimates in the Case A and Case B:

Case A: \(|\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+2\epsilon}{1+\epsilon}}\)

Case B: \(|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2} - \frac{3}{2}\epsilon}\)

If we take \(\epsilon = 0\), then

Case A: \(|\xi - \alpha| \ll H(\alpha)^{-n-1}\)

Case B: \(|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2}}\)

On the other hand, if we take \(\epsilon = 2\), then

Case A: \(|\xi - \alpha| \ll H(\alpha)^{-\frac{n+5}{3}}\)

Case B: \(|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - 4.5}\)

Finally, if we choose an optimal value of \(\epsilon\), namely

\[\epsilon = 1 - \frac{6}{n}\]

we obtain

\(|\xi - \alpha| \ll H(\alpha)^{-n/2 + \lambda_n}\), \(\lim_{n \to \infty} \lambda_n = 3\),

in both cases.
VI. “Improvement”

Let us consider an integer polynomial $P$ such that

\[ |a_n| \ll H, \ldots, |a_3| \ll H, \]
\[ |a_2| \ll H^{1+\epsilon}, \quad |a_1| \ll H^{1+\epsilon}, \quad |a_0| \ll H^{1+\epsilon}, \]
\[ |P(\xi)| \ll H^{-n-2\epsilon}. \]

We have

Case A: \[ |\xi - \alpha| \ll H(\alpha)^{-\frac{n+1+3\epsilon}{1+\epsilon}} \]

Case B: \[ |\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2} - \frac{3}{2} - 3\epsilon}. \]

Put \[ \epsilon = 1 - \frac{10}{n}, \]
then
\[ |\xi - \alpha| \ll H(\alpha)^{-n/2 + \lambda_n}, \quad \lim_{n \to \infty} \lambda_n = 4.5, \]
in both cases.

However, the Case A does not work. In fact,
\[ \max(|a_2|, |a_1|, |a_0|) \gg H \quad \Rightarrow \quad |P'(\xi)| \text{ is “big”}. \]
VII. Method of “Polynomial Staircase”

In 1996 a new approach to this problem was introduced:

Step 1. Let \( R^{(k)} \) be a polynomial with \( k \) “big” coefficients. We construct the following \( n \) polynomials

\[
Q^{(3)}, \ldots, Q^{(n+1)}, P^{(n+1)},
\]

which are small at \( \xi \).

Step 2. We prove that they are linearly independent.

Step 3. Using a linear combination of these polynomials, we construct the polynomial

\[
L^{(2)} = d_1 Q^{(3)} + \ldots + d_{n-1} Q^{(n+1)} + d_n P^{(n+1)}
\]

with two “big” coefficients. The Case A does work for \( L \). Moreover, it is possible to show that an influence of the numbers \( d_1, \ldots, d_n \) is very weak, so

\[
|L(\xi)| \ll H^{-n-2\epsilon}.
\]
Theorem 8. For any real number $\xi \not\in A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n}{2}} - \lambda_n, \quad \lim_{n \to \infty} \lambda_n = 4.$$
The following table contains the values of \( \frac{n}{2} + \lambda_n \)
corresponding to Wirsing’s Theorem (1961), the Theorem of Bernik-Tsishchanka (1993), Theorem 8 (2001), and the Conjecture:

<table>
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<th>( n )</th>
<th>1961</th>
<th>1993</th>
<th>2001</th>
<th>Conj.</th>
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<td>28.70</td>
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<tr>
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<td>51.99</td>
<td>52.92</td>
<td>53.84</td>
<td>101</td>
</tr>
</tbody>
</table>
VIII. Complex case

**Theorem 9 (Wirsing, 1961).** For any complex number $\xi \not\in A_n$ there exist infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \frac{n}{4} + 1.$$

**Method:** “Resultant”
In 2000 this result was slightly improved:

\[ A = \frac{n}{4} + \lambda_n, \quad \text{where} \quad \lim_{n \to \infty} \lambda_n = \frac{3}{2}. \]

**Method:** “Big Derivative”.

**Method “Polynomial Staircase”:** ? ? ?
IX. $P$-adic case

**Theorem 10** (Morrison, 1978). Let $\xi \in \mathbb{Q}_p$. If $\xi \notin A_n$, then there are infinitely many algebraic numbers $\alpha \in A_n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where

$$A = \begin{cases} 
1 + \sqrt{3} & \text{when } n = 2, \\
\frac{n}{2} + \frac{3}{2} & \text{when } n > 2. 
\end{cases}$$
Theorem 11 (Teulié, 2002). If \( \xi \not\in A_2 \), then there are infinitely many algebraic numbers \( \alpha \in A_2 \) with

\[
|\xi - \alpha| \ll H(\alpha)^{-3}.
\]

The second part of Morrison’s theorem was also improved:

\[
A = \frac{n}{2} + \lambda_n, \quad \text{where} \quad \lim_{n \to \infty} \lambda_n = 3.
\]

Method: “Big Derivative”.

Method “Polynomial Staircase”: ? ? ?
X. Two Counter-Examples

1. Simultaneous case.

Conjecture. For any two real numbers $\xi_1$, $\xi_2 \not\in A_n$ there exist infinitely many algebraic numbers $\alpha_1$, $\alpha_2$ such that

$$\begin{cases} |\xi_1 - \alpha_1| \ll |P|^{-(n+1)/2}, \\ |\xi_2 - \alpha_2| \ll |P|^{-(n+1)/2}, \end{cases}$$

where $P(x) \in \mathbb{Z}[x]$, $P(\alpha_1) = P(\alpha_2) = 0$, $\deg P \leq n$. The implicit constant in $\ll$ should depend on $\xi_1$, $\xi_2$, and $n$. 
**Counter-example** (Roy-Waldschmidt, 2001). For any sufficiently large $n$ there exist real numbers $\xi_1$ and $\xi_2$ such that

$$\max\{|\xi_1 - \alpha_1|, |\xi_2 - \alpha_2|\} > |P|^{-3\sqrt{n}}.$$ 

**Theorem 12.** For any real numbers $\xi_1, \xi_2 \notin A_n$ at least one of the following assertions is true:

(i) There exist infinitely many algebraic numbers $\alpha_1, \alpha_2$ of degree $\leq n$ such that

$$\begin{cases} 
|\xi_1 - \alpha_1| \ll |P|^{-\frac{n}{8} - \frac{3}{8}}, \\
|\xi_2 - \alpha_2| \ll |P|^{-\frac{n}{8} - \frac{3}{8}}.
\end{cases}$$

(ii) For some $\xi \in \{\xi_1, \xi_2\}$ there exist infinitely many algebraic numbers $\alpha$ of degree $2 \leq k \leq \frac{n+2}{4}$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-\frac{n+2}{4} - 1}.$$
2. Approximation by algebraic integers.

**Theorem 13.** (Davenport - Schmidt, 1968)

Let \( n \geq 3 \). Let \( \xi \) be real, but not algebraic of degree \( \leq 2 \). Then there are infinitely many algebraic integers \( \alpha \) of degree \( \leq 3 \) which satisfy

\[
0 < |\xi - \alpha| \ll H(\alpha)^{-\eta_3},
\]

where

\[
\eta_3 = \frac{1}{2} (3 + \sqrt{5}) = 2.618...\]
Conjecture. Let $\xi$ be real, but is not algebraic of degree $\leq n$. Suppose $\epsilon > 0$. Then there are infinitely many real algebraic integers $\alpha$ of degree $\leq n$ with

$$|\xi - \alpha| \ll H(\alpha)^{-n+\epsilon}.$$ 

Theorem 14 (Roy, 2001). There exist real numbers $\xi$ such that for any algebraic integer $\alpha$ of degree $\leq 3$, we have

$$|\xi - \alpha| \gg H(\alpha)^{-\eta_3}.$$
XI. Most Recent Result

**Theorem 15.** For any real number $\xi \not\in A_3$ there exist infinitely many algebraic numbers $\alpha \in A_3$ such that

$$|\xi - \alpha| \ll H(\alpha)^{-A},$$

where $A = 3.7475..$ is the largest root of the equation

$$2x^3 - 11x^2 + 11x + 8 = 0.$$