Example 1

Evaluate $\int_C x^4\,dx + xy\,dy$, where C is the triangular curve consisting of the line segments from (0, 0) to (1, 0), from (1, 0) to (0, 1), and from (0, 1) to (0, 0).

Solution:

Although the given line integral could be evaluated as usual by the methods, that would involve setting up three separate integrals along the three sides of the triangle, so let's use Green's Theorem instead.

Notice that the region D enclosed by C is simple and C has positive orientation (see Figure 4).

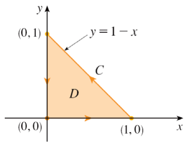

Figure 4

Example 1 – *Solution*

cont'd

If we let $P(x, y) = x^4$ and $Q(x, y) = xy$, then we have

$$\int_C x^4\,dx + xy\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

$$= \int_0^1 \int_0^{1-x} (y - 0)\,dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{2}\int_0^1 (1 - x)^2\,dx$$

$$= -\tfrac{1}{6}(1 - x)^3\Big]_0^1$$

$$= \tfrac{1}{6}$$

Example 4

Evaluate $\oint_C y^2\,dx + 3xy\,dy$, where C is the boundary of the semiannular region D in the upper half-plane between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 4$.

Solution:

Notice that although D is not simple, the y-axis divides it into two simple regions (see Figure 8).

In polar coordinates we can write

$D = \{(r, \theta) \mid 1 \le r \le 2,\ 0 \le \theta \le \pi\}$

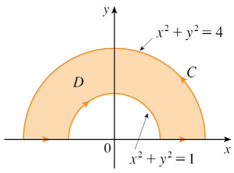

Figure 8

Example 4 – *Solution*

cont'd

Therefore Green's Theorem gives

$$\oint_C y^2\,dx + 3xy\,dy = \iint_D \left[\frac{\partial}{\partial x}(3xy) - \frac{\partial}{\partial y}(y^2)\right] dA$$

$$= \iint_D y\,dA$$

$$= \int_0^\pi \int_1^2 (r\sin\theta)\,r\,dr\,d\theta$$

$$= \int_0^\pi \sin\theta\,d\theta \int_1^2 r^2\,dr$$

$$= \big[-\cos\theta\big]_0^\pi \left[\tfrac{1}{3}r^3\right]_1^2 = \frac{14}{3}$$