Line Integrals

In this section we define an integral that is similar to a single integral except that instead of integrating over an interval $[a, b]$, we integrate over a curve C.

Such integrals are called *line integrals,* although “curve integrals” would be better terminology.

They were invented in the early 19th century to solve problems involving fluid flow, forces, electricity, and magnetism.

Line Integrals

We start with a plane curve C given by the parametric equations

$$\boxed{1} \qquad x = x(t) \qquad y = y(t) \qquad a \le t \le b$$

or, equivalently, by the vector equation $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$, and we assume that C is a smooth curve. [This means that $\mathbf{r}'$ is continuous and $\mathbf{r}'(t) \ne \mathbf{0}$.]

Line Integrals

If we divide the parameter interval $[a, b]$ into n subintervals $[t_{i-1}, t_i]$ of equal width and we let $x_i = x(t_i)$, and $y_i = y(t_i)$, then the corresponding points $P_i(x_i, y_i)$ divide C into n subarcs with lengths $\Delta s_1, \Delta s_2, \ldots, \Delta s_n$. (See Figure 1.)

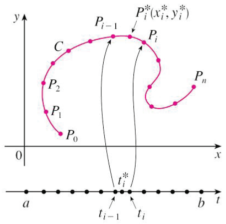

Figure 1

Line Integrals

We choose any point $P_i^*(x_i^*, y_i^*)$ in the ith subarc. (This corresponds to a point t_i^* in $[t_{i-1}, t_i]$.)

Now if f is any function of two variables whose domain includes the curve C, we evaluate f at the point (x_i^*, y_i^*), multiply by the length Δs_i of the subarc, and form the sum

$$\sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

which is similar to a Riemann sum.

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Then we take the limit of these sums and make the following definition by analogy with a single integral.

2 Definition If f is defined on a smooth curve C given by Equations 1, then the **line integral of f along C** is

$$\int_C f(x, y)\,ds = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*, y_i^*)\,\Delta s_i$$

if this limit exists.

We have found that the length of C is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

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A similar type of argument can be used to show that if f is a continuous function, then the limit in Definition 2 always exists and the following formula can be used to evaluate the line integral:

$$\boxed{3} \qquad \int_C f(x, y)\,ds = \int_a^b f(x(t), y(t))\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

The value of the line integral does not depend on the parametrization of the curve, provided that the curve is traversed exactly once as t increases from a to b.

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If $s(t)$ is the length of C between $\mathbf{r}(a)$ and $\mathbf{r}(t)$, then

$$\frac{ds}{dt} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}$$

So the way to remember Formula 3 is to express everything in terms of the parameter t: Use the parametric equations to express x and y in terms of t and write ds as

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

Line Integrals

In the special case where C is the line segment that joins $(a, 0)$ to $(b, 0)$, using x as the parameter, we can write the parametric equations of C as follows: $x = x$, $y = 0$, $a \le x \le b$.

Formula 3 then becomes

$$\int_C f(x, y)\,ds = \int_a^b f(x, 0)\,dx$$

and so the line integral reduces to an ordinary single integral in this case.

Line Integrals

Just as for an ordinary single integral, we can interpret the line integral of a *positive* function as an area.

In fact, if $f(x, y) \ge 0$, $\int_C f(x, y)\,ds$ represents the area of one side of the "fence" or "curtain" in Figure 2, whose base is C and whose height above the point (x, y) is $f(x, y)$.

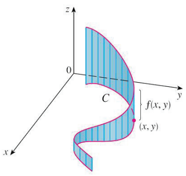


Figure 2

Example 1

Evaluate $\int_C (2 + x^2y)\,ds$, where C is the upper half of the unit circle $x^2 + y^2 = 1$.

Solution:

In order to use Formula 3, we first need parametric equations to represent C.

Recall that the unit circle can be parametrized by means of the equations

$x = \cos t \quad y = \sin t$

and the upper half of the circle is described by the parameter interval $0 \le t \le \pi$. (See Figure 3.)

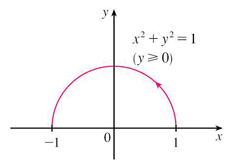


Figure 3

Example 1 – *Solution*

cont'd

Therefore Formula 3 gives

$$\int_C (2 + x^2y)\,ds = \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$= \int_0^\pi (2 + \cos^2 t \sin t)\sqrt{\sin^2 t + \cos^2 t}\,dt$$

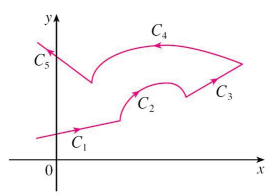
$$= \int_0^\pi (2 + \cos^2 t \sin t)\,dt$$

$$= \left[2t - \frac{\cos^3 t}{3}\right]_0^\pi$$

$$= 2\pi + \tfrac{2}{3}$$

Line Integrals

Suppose now that C is a **piecewise-smooth curve**; that is, C is a union of a finite number of smooth curves $C_1, C_2, \ldots, C_n$, where, as illustrated in Figure 4, the initial point of C_{i+1} is the terminal point of C_i.



A piecewise-smooth curve

Figure 4

Line Integrals

Then we define the integral of *f* along *C* as the sum of the integrals of *f* along each of the smooth pieces of *C*:

$$\int_C f(x, y)\, ds = \int_{C_1} f(x, y)\, ds + \int_{C_2} f(x, y)\, ds + \cdots + \int_{C_n} f(x, y)\, ds$$

Line Integrals

Any physical interpretation of a line integral $\int_C f(x, y)\, ds$ depends on the physical interpretation of the function *f*.

Suppose that $\rho(x, y)$ represents the linear density at a point (x, y) of a thin wire shaped like a curve *C*.

Then the mass of the part of the wire from P_{i-1} to P_i in Figure1 is approximately $\rho(x_i^*, y_i^*)\, \Delta s_i$ and so the total mass of the wire is approximately $\Sigma\, \rho(x_i^*, y_i^*)\, \Delta s_i$.

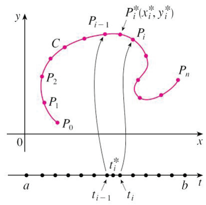


Figure 1

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By taking more and more points on the curve, we obtain the **mass** *m* of the wire as the limiting value of these approximations:

$$m = \lim_{n \to \infty} \sum_{i=1}^{n} \rho(x_i^*, y_i^*)\, \Delta s_i = \int_C \rho(x, y)\, ds$$

[For example, if $f(x, y) = 2 + x^2y$ represents the density of a semicircular wire, then the integral in Example 1 would represent the mass of the wire.]