Section 6.3 Applications

Averages

Average Values The ***average value*** of a function of one variable on the interval $[a, b]$ is defined by

$$[f]_{\mathrm{av}} = \frac{\int_a^b f(x)\,dx}{b-a}.$$

Likewise, for functions of two variables, the ratio of the integral to the area of D,

$$[f]_{\mathrm{av}} = \frac{\iint_D f(x,y)\,dx\,dy}{\iint_D dx\,dy}, \tag{1}$$

is called the ***average value*** of f over D. Similarly, the ***average value*** of a function f on a region W in three space is defined by

$$[f]_{\mathrm{av}} = \frac{\iiint_W f(x,y,z)\,dx\,dy\,dz}{\iiint_W dx\,dy\,dz}.$$

EXAMPLE: Find the average value of

$$f(x,y) = x\sin^2(xy)$$

on the region $D = [0,\pi]\times[0,\pi]$.

Solution: First, we compute

$$\begin{aligned}
\iint_D f(x,y)dxdy &= \int_0^\pi \int_0^\pi x\sin^2(xy)dxdy \\
&= \int_0^\pi \left[\int_0^\pi \frac{1-\cos(2xy)}{2} xdy\right] dx \\
&= \int_0^\pi \left[\frac{y}{2} - \frac{\sin(2xy)}{4x}\right] x\Big|_{y=0}^{\pi} dx \\
&= \int_0^\pi \left[\frac{\pi x}{2} - \frac{\sin(2\pi x)}{4x}\right] dx \\
&= \left[\frac{\pi x^2}{4} + \frac{\cos(2\pi x)}{8\pi}\right]\Big|_0^\pi \\
&= \frac{\pi^3}{4} + \frac{\cos(2\pi^2)-1}{8\pi}
\end{aligned}$$

Thus, the average value of f, by formula (1), is

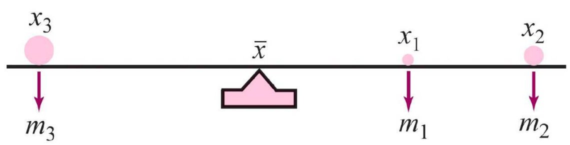
$$\frac{\pi^3/4 + [\cos(2\pi^2)-1]/8\pi}{\pi^2} = \frac{\pi}{4} + \frac{\cos(2\pi^2)-1}{8\pi^3} \approx 0.7839$$

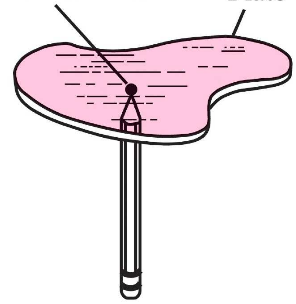
Centers of Mass

The Center of Mass of Two-Dimensional Plates

$$\bar{x} = \frac{\iint_D x\delta(x, y)\,dx\,dy}{\iint_D \delta(x, y)\,dx\,dy} \quad \text{and} \quad \bar{y} = \frac{\iint_D y\delta(x, y)\,dx\,dy}{\iint_D \delta(x, y)\,dx\,dy}, \tag{4}$$

where again $\delta(x, y)$ is the mass density (see Figure 6.3.2).





EXAMPLE 2 Find the mass and center of mass of a triangular lamina with vertices $(0, 0)$, $(1, 0)$, and $(0, 2)$ if the density function is $\rho(x, y) = 1 + 3x + y$.

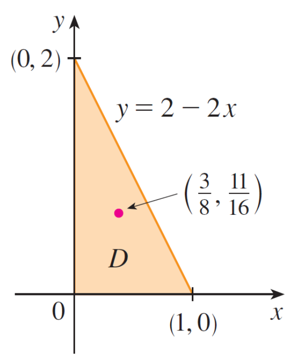


FIGURE 5

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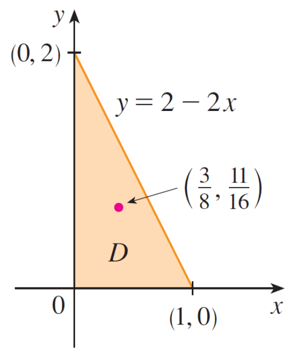


FIGURE 5

SOLUTION The triangle is shown in Figure 5. (Note that the equation of the upper boundary is $y = 2 - 2x$.) The mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \int_0^1 \int_0^{2-2x} (1 + 3x + y)\, dy\, dx$$

$$= \int_0^1 \left[y + 3xy + \frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx = 4 \int_0^1 (1 - x^2)\, dx = 4 \left[x - \frac{x^3}{3} \right]_0^1 = \frac{8}{3}$$

Then the formulas in (5) give

$$\bar{x} = \frac{1}{m} \iint_D x\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (x + 3x^2 + xy)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[xy + 3x^2y + x\frac{y^2}{2} \right]_{y=0}^{y=2-2x} dx = \tfrac{3}{2} \int_0^1 (x - x^3)\, dx$$

$$= \frac{3}{2} \left[\frac{x^2}{2} - \frac{x^4}{4} \right]_0^1 = \frac{3}{8}$$

$$\bar{y} = \frac{1}{m} \iint_D y\rho(x, y)\, dA = \tfrac{3}{8} \int_0^1 \int_0^{2-2x} (y + 3xy + y^2)\, dy\, dx$$

$$= \frac{3}{8} \int_0^1 \left[\frac{y^2}{2} + 3x\frac{y^2}{2} + \frac{y^3}{3} \right]_{y=0}^{y=2-2x} dx = \tfrac{1}{4} \int_0^1 (7 - 9x - 3x^2 + 5x^3)\, dx$$

$$= \frac{1}{4} \left[7x - 9\frac{x^2}{2} - x^3 + 5\frac{x^4}{4} \right]_0^1 = \frac{11}{16}$$

The center of mass is at the point $\left(\frac{3}{8}, \frac{11}{16}\right)$.

V EXAMPLE 3 The density at any point on a semicircular lamina is proportional to the distance from the center of the circle. Find the center of mass of the lamina.

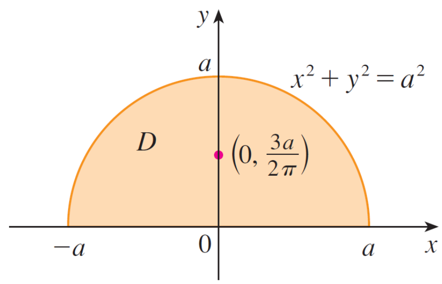


FIGURE 6

SOLUTION Let's place the lamina as the upper half of the circle $x^2 + y^2 = a^2$. (See Figure 6.) Then the distance from a point (x, y) to the center of the circle (the origin) is $\sqrt{x^2 + y^2}$. Therefore the density function is

$$\rho(x, y) = K\sqrt{x^2 + y^2}$$

where K is some constant. Both the density function and the shape of the lamina suggest that we convert to polar coordinates. Then $\sqrt{x^2 + y^2} = r$ and the region D is given by $0 \leqslant r \leqslant a$, $0 \leqslant \theta \leqslant \pi$. Thus the mass of the lamina is

$$m = \iint_D \rho(x, y)\, dA = \iint_D K\sqrt{x^2 + y^2}\, dA = \int_0^\pi \int_0^a (Kr)\, r\, dr\, d\theta$$

$$= K \int_0^\pi d\theta \int_0^a r^2\, dr = K\pi \frac{r^3}{3}\bigg]_0^a = \frac{K\pi a^3}{3}$$

Both the lamina and the density function are symmetric with respect to the y-axis, so the center of mass must lie on the y-axis, that is, $\bar{x} = 0$. The y-coordinate is given by

$$\bar{y} = \frac{1}{m} \iint_D y\, \rho(x, y)\, dA = \frac{3}{K\pi a^3} \int_0^\pi \int_0^a r \sin\theta\, (Kr)\, r\, dr\, d\theta$$

$$= \frac{3}{\pi a^3} \int_0^\pi \sin\theta\, d\theta \int_0^a r^3\, dr = \frac{3}{\pi a^3} \Big[-\cos\theta\Big]_0^\pi \left[\frac{r^4}{4}\right]_0^a$$

$$= \frac{3}{\pi a^3} \frac{2a^4}{4} = \frac{3a}{2\pi}$$

Therefore the center of mass is located at the point $(0, 3a/(2\pi))$.

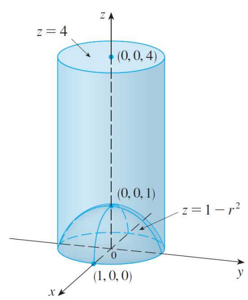
Coordinates for the Center of Mass of Three-Dimensional Regions

$$\bar{x} = \frac{\iiint_W x\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}},$$

$$\bar{y} = \frac{\iiint_W y\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}, \qquad (7)$$

$$\bar{z} = \frac{\iiint_W z\delta(x, y, z)\, dx\, dy\, dz}{\text{mass}}.$$

V EXAMPLE 3 A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. (See Figure 8.) The density at any point is proportional to its distance from the axis of the cylinder. Find the mass of E.



SOLUTION In cylindrical coordinates the cylinder is $r = 1$ and the paraboloid is $z = 1 - r^2$, so we can write

$$E = \left\{(r, \theta, z) \mid 0 \leqslant \theta \leqslant 2\pi,\ 0 \leqslant r \leqslant 1,\ 1 - r^2 \leqslant z \leqslant 4\right\}$$

Since the density at (x, y, z) is proportional to the distance from the z-axis, the density function is

$$f(x, y, z) = K\sqrt{x^2 + y^2} = Kr$$

where K is the proportionality constant. Therefore, from Formula 15.6.13, the mass of E is

$$m = \iiint_E K\sqrt{x^2 + y^2}\, dV$$

$$= \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (Kr)\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 Kr^2[4 - (1 - r^2)]\, dr\, d\theta$$

$$= K \int_0^{2\pi} d\theta \int_0^1 (3r^2 + r^4)\, dr$$

$$= 2\pi K\left[r^3 + \frac{r^5}{5}\right]_0^1 = \frac{12\pi K}{5}$$

Moments of Inertia

Another important concept in mechanics, one that is needed in studying the dynamics of a rotating rigid body, is that of *moment of inertia*. If the solid W has uniform density δ, the ***moments of inertia*** I_x, I_y, and I_z about the x, y, and z axes, respectively, are defined by:

Moments of Inertia About the Coordinate Axes

$$I_x = \iiint_W (y^2 + z^2)\,\delta\,dx\,dy\,dz, \qquad I_y = \iiint_W (x^2 + z^2)\,\delta\,dx\,dy\,dz,$$
$$I_z = \iiint_W (x^2 + y^2)\,\delta\,dx\,dy\,dz. \tag{8}$$

The moment of inertia measures a body's response to efforts to rotate it; for example, as when one tries to rotate a merry-go-round. The moment of inertia is analogous to the mass of a body, which measures its response to efforts to translate it. In contrast to translational motion, however, the moments of inertia *depend on the shape and not just the total mass*. It is harder to spin up a large plate than a compact ball of the same mass.

For example, I_x measures the body's response to forces attempting to rotate it about the x axis. The factor $y^2 + z^2$, which is the square of the distance to the x axis, weights masses farther away from the rotation axis more heavily. This is in agreement with the intuition just explained.

EXAMPLE: Compute the moment of inertia I_z for the solid above the xy plane bounded by the paraboloid $z = x^2 + y^2$ and the cylinder $x^2 + y^2 = a^2$, assuming a and the mass density to be constants.

Solution: The paraboloid and cylinder intersect at the plane $z = a^2$. Using cylindrical coordinates, we find from equation (8),

$$I_z = \int_0^a \int_0^{2\pi} \int_0^{r^2} \delta r^2 \cdot r dz d\theta dr = \delta \int_0^a \int_0^{2\pi} \int_0^{r^2} r^3 dz d\theta dr = \frac{\pi\delta a^6}{3}$$