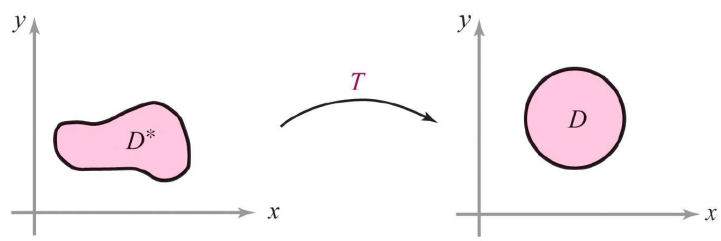
Section 6.1 The Geometry of Maps from $\mathbb{R}^2$ to $\mathbb{R}^2$

Maps of One Region to Another



EXAMPLE 1: Let $D^* \subset \mathbb{R}^2$ be the rectangle $D^* = [0,1] \times [0, 2\pi]$. Then all points on D^* are of the form (r, θ), where $0 \leq r \leq 1,\ 0 \leq \theta \leq 2\pi$. Let T be the polar coordinate "change of variables" defined by

$$T(r, \theta) = (r\cos\theta, r\sin\theta)$$

Find the image set D.

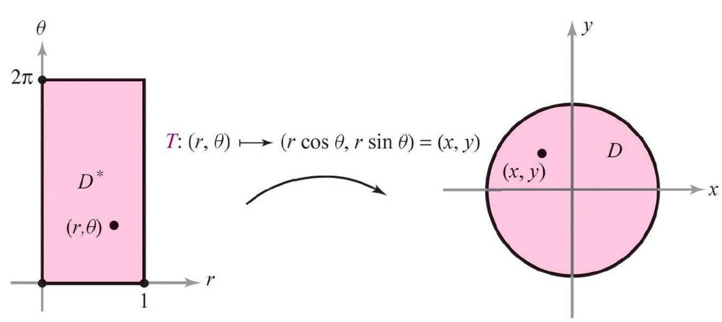
Solution: Let

$$(x, y) = (r\cos\theta, r\sin\theta)$$

Because of the identity

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2 \leq 1$$

we see that the set of points $(x, y) \in \mathbb{R}^2$ such that $(x, y) \in D$ has the property that $x^2 + y^2 \leq 1$, and so D is contained in the unit disk. In addition, any point (x, y) in the unit disk can be written as $(r\cos\theta, r\sin\theta)$ for some $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Thus, D is the unit disk.



EXAMPLE 2: Let T be defined by

$$T(x, y) = \left(\frac{x+y}{2}, \frac{x-y}{2}\right)$$

and let $D^* = [-1, 1] \times [-1, 1] \subset \mathbb{R}^2$ be a square with side of length 2 centered at the origin. Determine the image D obtained by applying T to D^*.

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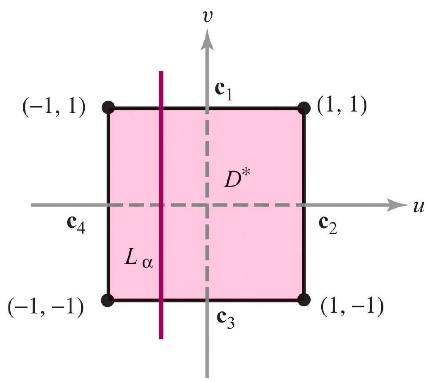
Solution: Let us first determine the effect of T on the line $\mathbf{c}_1(t) = (t,1)$, where $-1 \le t \le 1$.



We have

$$T(\mathbf{c}_1(t)) = \left(\frac{t+1}{2}, \frac{t-1}{2}\right)$$

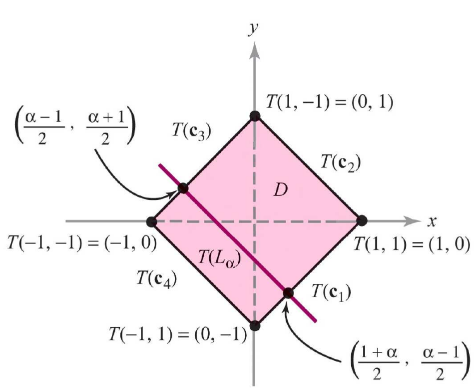
The map $t \mapsto T(\mathbf{c}_1(t))$ is a parametrization of the line $y = x - 1,\ 0 \le x \le 1$, because

$$\frac{t-1}{2} = \frac{t+1}{2} - 1$$

This is the straight line segment joining $(1,0)$ and $(0,-1)$. Let

$$\mathbf{c}_2(t) = (1,t), \quad -1 \le t \le 1$$
$$\mathbf{c}_3(t) = (t,-1), \quad -1 \le t \le 1$$
$$\mathbf{c}_4(t) = (-1,t), \quad -1 \le t \le 1$$

be parametrizations of the other edges of the square D^*. Using the same argument as before, we see that $T \circ \mathbf{c_2}$ is a parametrization of the line $y = 1 - x,\ 0 \le x \le 1$ (the straight line segment joining $(0,1)$ and $(1,0)$); $T \circ \mathbf{c_3}$ is the line $y = x + 1,\ -1 \le x \le 0$ joining $(0,1)$ and $(-1,0)$; and $T \circ \mathbf{c_4}$ is the line $y = -x - 1,\ -1 \le x \le 0$ joining $(-1,0)$ and $(0,-1)$. By this time seems reasonable to guess that T "flips" the square D^* over and takes it to the square D whose vertices are $(1,0), (0,1), (-1,0), (0,-1)$.



To prove that this is indeed the case, let $-1 \leq \alpha \leq 1$ and let L_α be a fixed line parametrized by $\mathbf{c}(t) = (\alpha, t),\ -1 \leq t \leq 1$; then

$$T(\mathbf{c}(t)) = \left(\frac{\alpha + t}{2}, \frac{\alpha - t}{2} \right)$$

is a parametrization of the line

$$y = -x + \alpha, \quad \frac{\alpha - 1}{2} \leq x \leq \frac{\alpha + 1}{2}$$

This line begins, for $t = -1$, at the point $((\alpha - 1)/2, (1 + \alpha)/2)$ and ends at the point $((1 + \alpha)/2, (\alpha - 1)/2)$; as is easily checked, these points lie on the lines $T \circ \mathbf{c_3}$ and $T \circ \mathbf{c}_1$, respectively. Thus, as α varies between -1 and 1, L_α sweeps out the square D^* while $T(L_\alpha)$ sweeps out the square D determined by the vertices $(-1, 0), (0, 1), (1, 0)$, and $(0, -1)$.

Images of Maps

The following theorem is a useful way to describe the image $T(D^*)$.

THEOREM 1 Let A be a 2×2 matrix with $\det A \neq 0$ and let T be the linear mapping of $\mathbb{R}^2$ to $\mathbb{R}^2$ given by $T(\mathbf{x}) = A\mathbf{x}$ (matrix multiplication). Then T transforms parallelograms into parallelograms and vertices into vertices. Moreover, if $T(D^*)$ is a parallelogram, D^* must be a parallelogram.

One-to-One Maps

DEFINITION A mapping T is ***one-to-one*** on D^* if for (u, v) and $(u', v') \in D^*$, $T(u, v) = T(u', v')$ implies that $u = u'$ and $v = v'$.

EXAMPLE 3: Consider the polar coordinate mapping function $T : \mathbb{R}^2 \to \mathbb{R}^2$ described in Example 1, defined by

$$T(r, \theta) = (r \cos \theta, r \sin \theta)$$

Show that T is not one-to one if its domain is all of $\mathbb{R}^2$.

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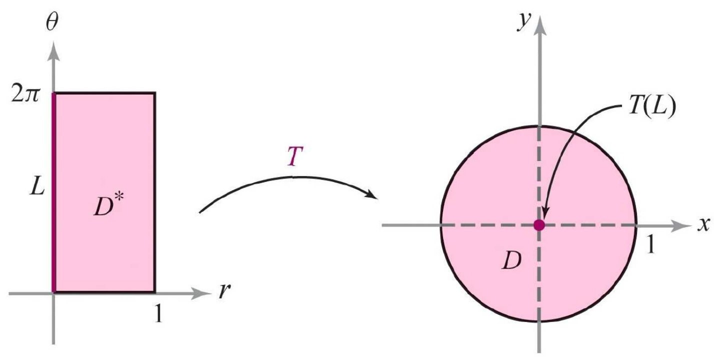
$$T(r, \theta) = (r \cos\theta, r \sin\theta)$$

Show that T is not one-to one if its domain is all of $\mathbb{R}^2$.

Solution: If $\theta_1 \neq \theta_2$, then $T(0, \theta_1) = T(0, \theta_2)$, and so T cannot be one-to-one. This observation implies that if L is the side of the rectangle

$$D^* = [0, 1] \times [0, 2\pi]$$

where $0 \leq \theta \leq 2\pi$ and $r = 0$, then T maps all of L into a single point, the center of the unit disk D.



However, if we consider the set

$$S^* = (0, 1] \times [0, 2\pi)$$

then $T : S^* \to S$ is one-to-one (see Exercise 1). Evidently, in determining whether a function is one-to-one, the domain chosen must be carefully considered.

EXAMPLE 4: Show that the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ of Example 2 is one-to-one.

Solution: Suppose $T(x, y) = T(x', y')$; then

$$\left(\frac{x+y}{2}, \frac{x-y}{2}\right) = \left(\frac{x'+y'}{2}, \frac{x'-y'}{2}\right)$$

and we have

$$x + y = x' + y'$$

$$x - y = x' - y'$$

Adding, we have

$$2x = 2x'$$

Thus, $x = x'$ and, similarly, subtracting gives $y = y'$, which shows that T is one-to-one (with domain all of $\mathbb{R}^2$). Actually, because T is linear and $T(\mathbf{x}) = A\mathbf{x}$, where A is a 2×2 matrix, it would also suffice to show that $\det A \neq 0$.

Onto Maps

DEFINITION The mapping T is ***onto*** D if for every point $(x, y) \in D$ there exists at least one point (u, v) in the domain of T such that $T(u, v) = (x, y)$.

EXAMPLE 5: Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be given by

$$T(u, v) = (u, 0)$$

Let D be the square, $D = [0, 1] \times [0, 1]$. Because T takes all of $\mathbb{R}^2$ to one axis, it is impossible to find a D^* such that $T(D^*) = D$.

EXAMPLE 6: Let T be defined as in Example 2 and let D be the square whose vertices are $(1, 0), (0, 1), (-1, 0), (0, -1)$. Find a D^* with $T(D^*) = D$.

Solution: Because T is linear and $T(\mathbf{x}) = A(\mathbf{x})$, where A is a 2×2 matrix satisfying $\det A \neq 0$, we know that $T : \mathbb{R}^2 \to \mathbb{R}^2$ is onto (see Exercises 8 and 9), and thus D^* can be found. By Theorem 1, D^* must be a parallelogram. In order to find D^*, it suffices to find the four points that are mapped onto vertices of D; then, by connecting these points, we will have found D^*. For the vertex $(1, 0)$ of D, we must solve

$$T(x, y) = (1, 0) = \left(\frac{x + y}{2}, \frac{x - y}{2} \right)$$

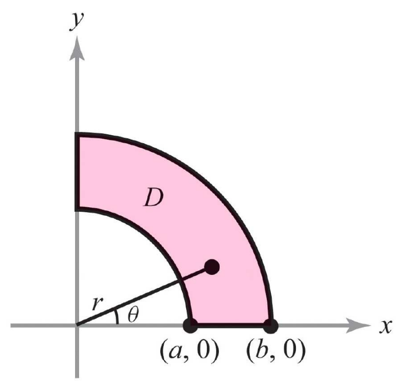
so that

$$\frac{x + y}{2} = 1, \quad \frac{x - y}{2} = 0$$

Thus, $(x, y) = (1, 1)$ is a vertex of D^*. Solving for the other vertices, we find that $D^* = [-1, 1] \times [-1, 1]$. This is in agreement with what we found more laboriously in Example 2.

EXAMPLE 7: Let D be the region in the first quadrant lying between the arcs of the circles

$$x^2 + y^2 = a^2, \quad x^2 + y^2 = b^2, \quad 0 < a < b$$



These circles have equations $r = a$ and $r = b$ in polar coordinates. Let T be the polar coordinate transformation given by

$$T(r, \theta) = (r \cos\theta, r \sin\theta) = (x, y)$$

Find D^* such that $T(D^*) = D$.

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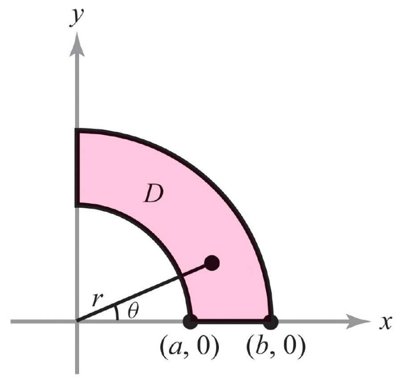

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Find D^* such that $T(D^*) = D$.

Solution: In the region $D, a^2 \leq x^2 + y^2 \leq b^2$; and because $r^2 = x^2 + y^2$, we see that $a \leq r \leq b$. Clearly, for this region θ varies between $0 \leq \theta \leq \pi/2$. Thus, if

$$D^* = [a, b] \times [0, \pi/2]$$

we have

$$T(D^*) = D$$

and T is one-to-one.

One-to-One and Onto Mappings A mapping $T: D^* \to D$ is ***one-to-one*** when it maps distinct points to distinct points. It is ***onto*** when the image of D^* under T is all of D.

A *linear* transformation of $\mathbb{R}^n$ to $\mathbb{R}^n$ given by multiplication by a matrix A is one-to-one and onto when and only when $\det A \neq 0$.