

## Iterated Integrals

Suppose that  $f$  is a function of two variables that is integrable on the rectangle  $R = [a, b] \times [c, d]$ .

We use the notation  $\int_c^d f(x, y) dy$  to mean that  $x$  is held fixed and  $f(x, y)$  is integrated with respect to  $y$  from  $y = c$  to  $y = d$ . This procedure is called *partial integration with respect to  $y$* . (Notice its similarity to partial differentiation.)

Now  $\int_c^d f(x, y) dy$  is a number that depends on the value of  $x$ , so it defines a function of  $x$ :

$$A(x) = \int_c^d f(x, y) dy$$

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If we now integrate the function  $A$  with respect to  $x$  from  $x = a$  to  $x = b$ , we get

$$\boxed{1} \quad \int_a^b A(x) dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

The integral on the right side of Equation 1 is called an **iterated integral**. Usually the brackets are omitted. Thus

$$\boxed{2} \quad \int_a^b \int_c^d f(x, y) dy dx = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx$$

means that we first integrate with respect to  $y$  from  $c$  to  $d$  and then with respect to  $x$  from  $a$  to  $b$ .

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Similarly, the iterated integral

$$\boxed{3} \quad \int_c^d \int_a^b f(x, y) dx dy = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy$$

means that we first integrate with respect to  $x$  (holding  $y$  fixed) from  $x = a$  to  $x = b$  and then we integrate the resulting function of  $y$  with respect to  $y$  from  $y = c$  to  $y = d$ .

Notice that in both Equations 2 and 3 we work *from the inside out*.

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## Example 1

Evaluate the iterated integrals.

$$(a) \int_0^3 \int_1^2 x^2 y dy dx \quad (b) \int_1^2 \int_0^3 x^2 y dx dy$$

**Solution:**

(a) Regarding  $x$  as a constant, we obtain

$$\begin{aligned} \int_1^2 x^2 y dy &= \left[ x^2 \frac{y^2}{2} \right]_{y=1}^{y=2} \\ &= x^2 \left( \frac{2^2}{2} \right) - x^2 \left( \frac{1^2}{2} \right) \\ &= \frac{3}{2} x^2 \end{aligned}$$

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## Example 1 – Solution

cont'd

Thus the function  $A$  in the preceding discussion is given by  $A(x) = \frac{3}{2}x^2$  in this example.

We now integrate this function of  $x$  from 0 to 3:

$$\begin{aligned} \int_0^3 \int_1^2 x^2 y dy dx &= \int_0^3 \left[ \int_1^2 x^2 y dy \right] dx \\ &= \int_0^3 \frac{3}{2} x^2 dx \\ &= \left[ \frac{x^3}{2} \right]_0^3 \\ &= \frac{27}{2} \end{aligned}$$

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## Example 1 – Solution

cont'd

(b) Here we first integrate with respect to  $x$ :

$$\begin{aligned} \int_1^2 \int_0^3 x^2 y dx dy &= \int_1^2 \left[ \int_0^3 x^2 y dx \right] dy \\ &= \int_1^2 \left[ \frac{x^3}{3} y \right]_{x=0}^{x=3} dy \\ &= \int_1^2 9y dy \\ &= 9 \left[ \frac{y^2}{2} \right]_1^2 = \frac{27}{2} \end{aligned}$$

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Notice that in Example 1 we obtained the same answer whether we integrated with respect to  $y$  or  $x$  first.

In general, it turns out (see Theorem 4) that the two iterated integrals in Equations 2 and 3 are always equal; that is, the order of integration does not matter. (This is similar to Clairaut's Theorem on the equality of the mixed partial derivatives.)

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## Iterated Integrals

The following theorem gives a practical method for evaluating a double integral by expressing it as an iterated integral (in either order).

**4 Fubini's Theorem** If  $f$  is continuous on the rectangle

$R = \{(x, y) \mid a \leq x \leq b, c \leq y \leq d\}$ , then

$$\iint_R f(x, y) \, dA = \int_a^b \int_c^d f(x, y) \, dy \, dx = \int_c^d \int_a^b f(x, y) \, dx \, dy$$

More generally, this is true if we assume that  $f$  is bounded on  $R$ ,  $f$  is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

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## Iterated Integrals

In the special case where  $f(x, y)$  can be factored as the product of a function of  $x$  only and a function of  $y$  only, the double integral of  $f$  can be written in a particularly simple form.

To be specific, suppose that  $f(x, y) = g(x)h(y)$  and  $R = [a, b] \times [c, d]$ .

Then Fubini's Theorem gives

$$\iint_R f(x, y) \, dA = \int_c^d \int_a^b g(x)h(y) \, dx \, dy = \int_c^d \left[ \int_a^b g(x)h(y) \, dx \right] dy$$

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In the inner integral,  $y$  is a constant, so  $h(y)$  is a constant and we can write

$$\int_c^d \left[ \int_a^b g(x)h(y) \, dx \right] dy = \int_c^d \left[ h(y) \left( \int_a^b g(x) \, dx \right) \right] dy = \int_a^b g(x) \, dx \int_c^d h(y) \, dy$$

since  $\int_a^b g(x) \, dx$  is a constant.

Therefore, in this case, the double integral of  $f$  can be written as the product of two single integrals:

$$\iint_R g(x)h(y) \, dA = \int_a^b g(x) \, dx \int_c^d h(y) \, dy \quad \text{where } R = [a, b] \times [c, d]$$

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