Curl

Curl

If $\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$ is a vector field on $\mathbb{R}^3$ and the partial derivatives of P, Q, and R all exist, then the **curl** of $\mathbf{F}$ is the vector field on $\mathbb{R}^3$ defined by

$$\boxed{1} \qquad \operatorname{curl} \mathbf{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

Let's rewrite Equation 1 using operator notation. We introduce the vector differential operator ∇ ("del") as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

Curl

It has meaning when it operates on a scalar function to produce the gradient of f:

$$\nabla f = \mathbf{i}\frac{\partial f}{\partial x} + \mathbf{j}\frac{\partial f}{\partial y} + \mathbf{k}\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

If we think of ∇ as a vector with components $\partial/\partial x$, $\partial/\partial y$, and $\partial/\partial z$, we can also consider the formal cross product of ∇ with the vector field $\mathbf{F}$ as follows:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

Curl

$$= \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\mathbf{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\mathbf{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}$$

$$= \operatorname{curl} \mathbf{F}$$

So the easiest way to remember Definition 1 is by means of the symbolic expression

$$\boxed{2} \qquad \boxed{\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F}}$$

Example 1

If $\mathbf{F}(x, y, z) = xz\,\mathbf{i} + xyz\,\mathbf{j} - y^2\,\mathbf{k}$, find curl $\mathbf{F}$.

Solution:

Using Equation 2, we have

$$\operatorname{curl} \mathbf{F} = \nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xz & xyz & -y^2 \end{vmatrix}$$

Example 1 – *Solution*

cont'd

$$= \left[\frac{\partial}{\partial y}(-y^2) - \frac{\partial}{\partial z}(xyz)\right]\mathbf{i} - \left[\frac{\partial}{\partial x}(-y^2) - \frac{\partial}{\partial z}(xz)\right]\mathbf{j}$$

$$+ \left[\frac{\partial}{\partial x}(xyz) - \frac{\partial}{\partial y}(xz)\right]\mathbf{k}$$

$$= (-2y - xy)\,\mathbf{i} - (0 - x)\,\mathbf{j} + (yz - 0)\,\mathbf{k}$$

$$= -y(2 + x)\,\mathrm{i} + x\,\mathbf{j} + yz\,\mathbf{k}$$

Curl

Recall that the gradient of a function *f* of three variables is a vector field on $\mathbb{R}^3$ and so we can compute its curl.

The following theorem says that the curl of a gradient vector field is **0**.

> **3 Theorem** If f is a function of three variables that has continuous second-order partial derivatives, then
>
> $$\operatorname{curl}(\nabla f) = \mathbf{0}$$

Curl

Since a conservative vector field is one for which $\mathbf{F} = \nabla f$, Theorem 3 can be rephrased as follows:

If **F** is conservative, then curl **F** = **0**.

This gives us a way of verifying that a vector field is not conservative.

Curl

The converse of Theorem 3 is not true in general, but the following theorem says the converse is true if **F** is defined everywhere. (More generally it is true if the domain is simply-connected, that is, "has no hole.")

> **4 Theorem** If **F** is a vector field defined on all of $\mathbb{R}^3$ whose component functions have continuous partial derivatives and curl $\mathbf{F} = \mathbf{0}$, then **F** is a conservative vector field.

Curl

The reason for the name *curl* is that the curl vector is associated with rotations.

Another occurs when **F** represents the velocity field in fluid flow. Particles near (*x*, *y*, *z*) in the fluid tend to rotate about the axis that points in the direction of curl **F**(*x*, *y*, *z*), and the length of this curl vector is a measure of how quickly the particles move around the axis (see Figure 1).

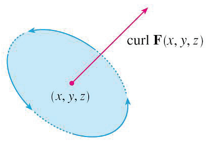

Figure 1

Curl

If curl **F** = **0** at a point *P*, then the fluid is free from rotations at *P* and **F** is called **irrotational** at *P*.

In other words, there is no whirlpool or eddy at *P*.

If curl **F** = **0**, then a tiny paddle wheel moves with the fluid but doesn't rotate about its axis.

If curl **F** ≠ **0**, the paddle wheel rotates about its axis.

Divergence

Divergence

If **F** = P **i** + Q **j** + R **k** is a vector field on $\mathbb{R}^3$ and $\partial P/\partial x$, $\partial Q/\partial y$, and $\partial R/\partial z$ exist, then the **divergence of F** is the function of three variables defined by

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$$\text{div } \mathbf{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Observe that curl **F** is a vector field but div **F** is a scalar field.

Divergence

In terms of the gradient operator $\nabla = (\partial/\partial x)\, \mathbf{i} + (\partial/\partial y)\, \mathbf{j} + (\partial/\partial z)\, \mathbf{k}$, the divergence of **F** can be written symbolically as the dot product of ∇ and **F**:

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$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

Example 4

If $\mathbf{F}(x, y, z) = xz\, \mathbf{i} + xyz\, \mathbf{j} - y^2\, \mathbf{k}$, find div **F**.

Solution:

By the definition of divergence (Equation 9 or 10) we have

$$\text{div } \mathbf{F} = \nabla \cdot \mathbf{F}$$

$$= \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2)$$

$$= z + xz$$

Divergence

If **F** is a vector field on $\mathbb{R}^3$, then curl **F** is also a vector field on $\mathbb{R}^3$. As such, we can compute its divergence.

The next theorem shows that the result is 0.

11 Theorem If $\mathbf{F} = P\, \mathbf{i} + Q\, \mathbf{j} + R\, \mathbf{k}$ is a vector field on $\mathbb{R}^3$ and P, Q, and R have continuous second-order partial derivatives, then

$$\text{div curl } \mathbf{F} = 0$$

Again, the reason for the name *divergence* can be understood in the context of fluid flow.

Divergence

If **F**(x, y, z) is the velocity of a fluid (or gas), then div **F**(x, y, z) represents the net rate of change (with respect to time) of the mass of fluid (or gas) flowing from the point (x, y, z) per unit volume.

In other words, div **F**(x, y, z) measures the tendency of the fluid to diverge from the point (x, y, z).

If div **F** = 0, then **F** is said to be **incompressible**.

Another differential operator occurs when we compute the divergence of a gradient vector field ∇f.

Divergence

If f is a function of three variables, we have

$$\text{div}(\nabla f) = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

and this expression occurs so often that we abbreviate it as $\nabla^2 f$. The operator

$$\nabla^2 = \nabla \cdot \nabla$$

is called the **Laplace operator** because of its relation to **Laplace's equation**

$$\nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = 0$$

Divergence

We can also apply the Laplace operator ∇^2 to a vector field

$$\mathbf{F} = P\,\mathbf{i} + Q\,\mathbf{j} + R\,\mathbf{k}$$

in terms of its components:

$$\nabla^2\mathbf{F} = \nabla^2 P\,\mathbf{i} + \nabla^2 Q\,\mathbf{j} + \nabla^2 R\,\mathbf{k}$$