Section 4.4 Divergence and Curl

We are now about to take derivatives of a vector field. Since the vector field have three components and each depends on three variables, we are faced with trying to interpret the meaning of nine different derivatives. However, it turns out that certain combinations of these derivatives have a clear geometric and physical meaning. One is the *divergence* of a vector field which is a scalar field and the other is the *curl* of a vector field which is a vector field. The divergence tells us to what extent the field is spreading the particles out, "diverging". The curl tells us how the vector field "swirls" particles around.

For each of the divergence and curl operations, we will make use of the *del operator*, defined by

$$\boxed{\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}}$$

DEFINITION: If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the *divergence* of $\mathbf{F}$ is the scalar field

$$\boxed{\text{div } \mathbf{F} = \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}}$$

EXAMPLES:

1. If $\mathbf{F} = x\mathbf{i} + y\mathbf{j}$, then div $\mathbf{F} = 2$.

2. If $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$, then div $\mathbf{F} = 0$.

From these two examples it appears that the description of divergence as how much the flow lines go apart is correct. However, there is more to it:

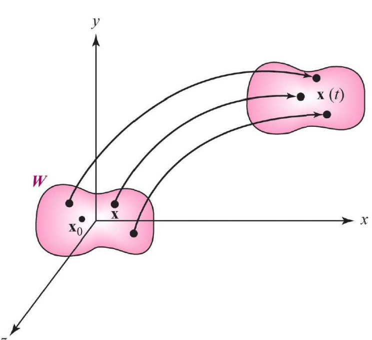
EXAMPLE: Let

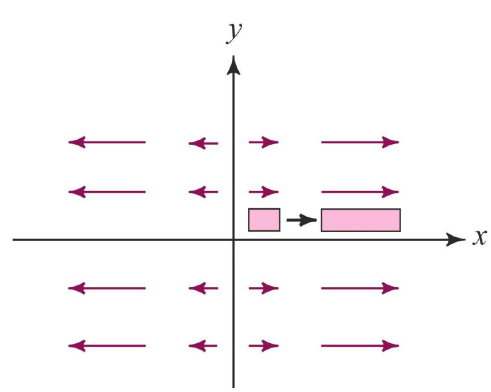
$$\mathbf{F} = \frac{x\mathbf{i} + y\mathbf{j}}{x^2 + y^2}$$

then div $\mathbf{F} = 0$.

The flow lines in the last example are the same as in example 1 since only the magnitude of the vector field changed, i.e., by example 1, lines go out from the origin. The magnitude of the vector field in the last example however decreases as we go out, and that compensates for that the flow lines go apart, and it makes the divergence smaller.

Physical interpretation of the divergence





The divergence of the velocity vector field of a fluid is the rate of expansion of the fluid per unit volume. If $\mathbf{V}$ is the velocity vector field of the fluid and $\mathbf{x}$ is the position of a fluid particle, then

$$\frac{d\mathbf{x}}{dt} = \mathbf{V}(\mathbf{x})$$

If we follow the positions of all fluid particles within a small domain, we get a domain $\mathcal{D}_t$ depending on time t. The rate of change of the volume $\mathrm{Vol}(\mathcal{D}_t)$ of this domain is

$$\frac{d\mathrm{Vol}(\mathcal{D}_t)}{dt} = \mathrm{Vol}(\mathcal{D}_t)\mathrm{div}\mathbf{V}$$

i.e. the divergence is the *rate of expansion* of the fluid volume per unit volume. An incompressible liquid is divergence free $\mathrm{div}\mathbf{V} = 0$ whereas a gas is compressible and the divergence is nonvanishing.

If the fluid expands, then the average density must decrease and fluid must flow out of any fixed region. If we instead consider the amount of fluid $\mathcal{M}_t$ in a small fixed domain $\mathcal{D}$ and define the *flow rate density* by

$$\mathbf{F} = \mu\mathbf{V}$$

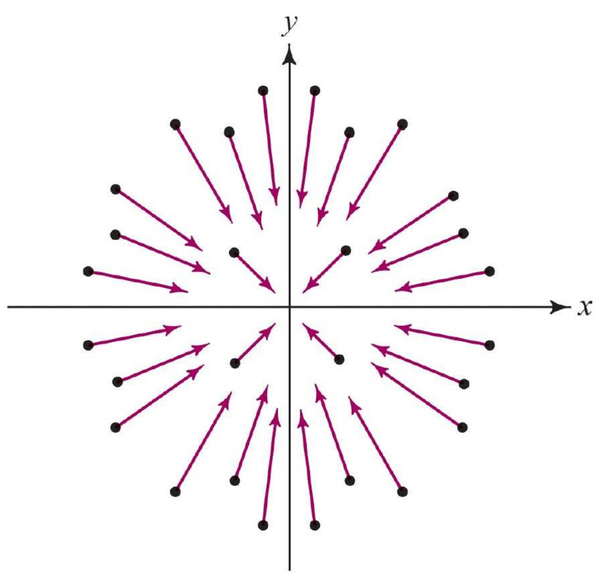
where μ is the density, then the *rate of increase* of the amount of fluid in $\mathcal{D}$ is

$$\frac{d\mathcal{M}_t}{dt} = -\mathrm{Vol}(\mathcal{D})\mathrm{div}\mathbf{F}$$

The rate of change of the amount of fluid in the domain is equal to the amount of fluid that goes out through the surface S of the domain per unit time.

EXAMPLES:

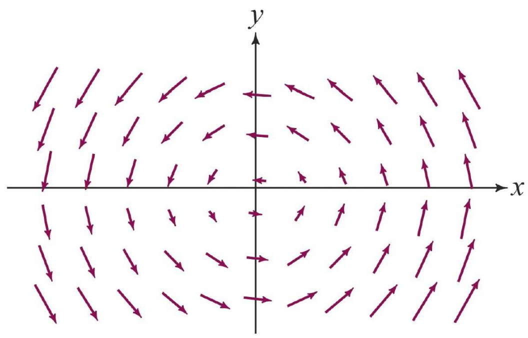
1. Consider the vector field $\mathbf{F} = -x\mathbf{i} - y\mathbf{j}$. Here the flow lines point toward the origin instead of away from it:



Therefore, the fluid is compressing, so we expect (div $\mathbf{F} < 0$). Calculating, we see that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-x) + \frac{\partial}{\partial y}(-y) = -1 - 1 = -2 < 0$$

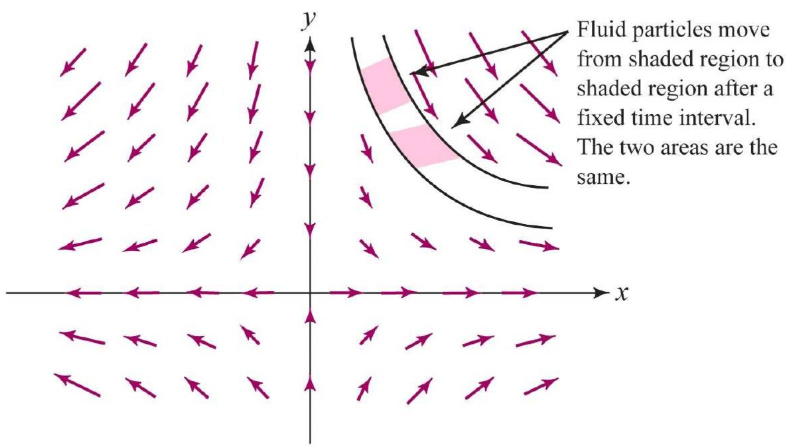
2. As we saw in the last section, the flow lines of $\mathbf{F} = -y\mathbf{i} + x\mathbf{j}$ are concentric circles about the origin, moving counterclockwise:



From this figure, it appears that the fluid is neither compressing nor expanding. This is confirmed by calculating

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0 + 0 = 0$$

3. Some flow lines of $\mathbf{F} = x\mathbf{i} - y\mathbf{j}$ are shown below:



Here our intuition about expansion or compression is less clear. However, it is true that the shaded regions shown have the same area, and we calculate that

$$\nabla \cdot \mathbf{F} = \frac{\partial}{\partial x}x + \frac{\partial}{\partial y}(-y) = 1 + (-1) = 0$$

Curl

DEFINITION: If $\mathbf{F} = F_1\mathbf{i} + F_2\mathbf{j} + F_3\mathbf{k}$, the <u>*curl*</u> of $\mathbf{F}$ is the *vector field*

$$\boxed{\begin{aligned} \text{curl } \mathbf{F} = \nabla \times \mathbf{F} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix} \\ &= \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right)\mathbf{i} + \left(\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x}\right)\mathbf{j} + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right)\mathbf{k} \end{aligned}}$$

EXAMPLES:

1. Let $\mathbf{F}(x, y, z) = x\mathbf{i} + xy\mathbf{j} + \mathbf{k}$. Find $\nabla \times \mathbf{F}$.

Solution: We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xy & 1 \end{vmatrix} = (0-0)\,\mathbf{i} - (0-0)\,\mathbf{j} + (y-0)\,\mathbf{k} = y\mathbf{k}$$

2. Find the curl of $xy\mathbf{i} - \sin z\mathbf{j} + \mathbf{k}$.

Solution: We have

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy & -\sin z & 1 \end{vmatrix} = \cos z\mathbf{i} - x\mathbf{k}$$

Gradients are Curl Free

THEOREM (Curl of a Gradient): For any C^2 function f,

$$\boxed{\nabla \times (\nabla f) = \mathbf{0}}$$

That is, the curl of any gradient is the zero vector.

Proof: Because

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

we have, by definition,

$$\nabla \times \nabla f = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{vmatrix} = \left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right)\mathbf{i} + \left(\frac{\partial^2 f}{\partial z \partial x} - \frac{\partial^2 f}{\partial x \partial z}\right)\mathbf{j} + \left(\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial z}\right)\mathbf{k}$$

Each component is zero because of the equality of mixed partial derivatives. ■

EXAMPLE: Let $\mathbf{V}(x, y, z) = y\mathbf{i} - x\mathbf{j}$. Show that $\mathbf{V}$ is not a gradient field.

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Solution: If $\mathbf{V}$ were a gradient field, then it would satisfy curl $\mathbf{V} = \mathbf{0}$ by the above Theorem. But

$$\operatorname{curl} \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\mathbf{k} \neq \mathbf{0}$$

so $\mathbf{V}$ cannot be a gradient.

Scalar Curl

If $\mathbf{F} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ is a vector field in the plane, it can also be regarded as a vector field in space for which the $\mathbf{k}$ component is zero and the other two components are independent of z. The curl of $\mathbf{F}$ the reduces to

$$\boxed{\nabla \times \mathbf{F} = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\mathbf{k}}$$

and always points in the $\mathbf{k}$ direction. The function

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}$$

of x and y is called the *scalar curl* of $\mathbf{F}$.

EXAMPLE: Find the scalar curl of $\mathbf{V}(x, y) = -y^2\mathbf{i} + x\mathbf{j}$.

Solution: The curl is

$$\nabla \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ -y^2 & x & 0 \end{vmatrix} = (1 + 2y)\mathbf{k}$$

so the scalar curl, which is the coefficient of $\mathbf{k}$, is $1 + 2y$.

Curls are Divergence Free

THEOREM (Divergence of a Curl): For any C^2 vector field $\mathbf{F}$,

$$\boxed{\operatorname{div} \operatorname{curl} \mathbf{F} = \nabla \cdot (\nabla \times \mathbf{F}) = 0}$$

That is, the divergence of any curl is zero.

EXAMPLE: Show that the vector field $\mathbf{V}(x, y, z) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ cannot be the curl of some vector field $\mathbf{F}$; that is, there is no $\mathbf{F}$ with $\mathbf{V} = \operatorname{curl} \mathbf{F}$.

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Solution: If this were so, then div $\mathbf{V}$ would be zero by the above Theorem. But

$$\text{div } \mathbf{V} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3 \neq 0$$

so $\mathbf{V}$ cannot be curl $\mathbf{F}$ for any $\mathbf{F}$.

Laplacian

The *Laplace operator* ∇^2, which operates on functions f, is defined to be the divergence of the gradient:

$$\boxed{\nabla^2 f = \nabla \cdot (\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}}$$

EXAMPLE: Show that $\nabla^2 f = 0$ for

$$f(x, y, z) = \frac{1}{\sqrt{x^2 + y^2 + z^2}} = \frac{1}{r} \quad \text{and} \quad (x, y, z) \neq (0, 0, 0)$$

where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $r = \|\mathbf{r}\|$.

Solution: We have

$$\frac{\partial f}{\partial x} = \frac{-x}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \frac{\partial f}{\partial y} = \frac{-y}{(x^2 + y^2 + z^2)^{3/2}}, \qquad \frac{\partial f}{\partial z} = \frac{-z}{(x^2 + y^2 + z^2)^{3/2}}$$

Therefore

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} &= \frac{3x^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{\partial^2 f}{\partial y^2} &= \frac{3y^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}} \\
\frac{\partial^2 f}{\partial z^2} &= \frac{3z^2}{(x^2 + y^2 + z^2)^{5/2}} - \frac{1}{(x^2 + y^2 + z^2)^{3/2}}
\end{aligned}$$

Thus,

$$\begin{aligned}
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} &= \frac{3(x^2 + y^2 + z^2)}{(x^2 + y^2 + z^2)^{5/2}} - \frac{3}{(x^2 + y^2 + z^2)^{3/2}} \\
&= \frac{3}{(x^2 + y^2 + z^2)^{3/2}} - \frac{3}{(x^2 + y^2 + z^2)^{3/2}} = 0
\end{aligned}$$

Vector Identities

1. $\nabla(f+g) = \nabla f + \nabla g$
2. $\nabla(cf) = c\nabla f$, for a constant c
3. $\nabla(fg) = f\nabla g + g\nabla f$
4. $\nabla(f/g) = (g\nabla f - f\nabla g)/g^2$, at points $\mathbf{x}$ where $g(\mathbf{x}) \neq 0$
5. div $(\mathbf{F}+\mathbf{G})$ = div $\mathbf{F}$+ div $\mathbf{G}$
6. curl $(\mathbf{F}+\mathbf{G})$ = curl $\mathbf{F}$+ curl $\mathbf{G}$
7. div $(f\mathbf{F}) = f$div $\mathbf{F} + \mathbf{F} \cdot \nabla f$
8. div $(\mathbf{F} \times \mathbf{G}) = \mathbf{G}\cdot$curl $\mathbf{F} - \mathbf{F}\cdot$curl $\mathbf{G}$
9. div curl $\mathbf{F} = 0$
10. curl $(f\mathbf{F}) = f$curl $\mathbf{F} + \nabla f \times \mathbf{F}$
11. curl $\nabla f = \mathbf{0}$
12. $\nabla^2(fg) = f\nabla^2 g + g\nabla^2 f + 2(\nabla f \cdot \nabla g)$
13. div $(\nabla f \times \nabla g) = 0$
14. div $(f\nabla g - g\nabla f) = f\nabla^2 g - g\nabla^2 f$