Theorem 11 Special Implicit Function Theorem Suppose that $F: \mathbb{R}^{n+1} \to \mathbb{R}$ has continuous partial derivatives. Denoting points in $\mathbb{R}^{n+1}$ by $(\mathbf{x}, z)$, where $\mathbf{x} \in \mathbb{R}^n$ and $z \in \mathbb{R}$, assume that $(\mathbf{x}_0, z_0)$ satisfies

$$F(\mathbf{x}_0, z_0) = 0 \quad \text{and} \quad \frac{\partial F}{\partial z}(\mathbf{x}_0, z_0) \neq 0.$$

Then there is a ball U containing $\mathbf{x}_0$ in $\mathbb{R}^n$ and a neighborhood V of z_0 in $\mathbb{R}$ such that there is a unique function $z = g(\mathbf{x})$ defined for $\mathbf{x}$ in U and z in V that satisfies

$$F(\mathbf{x}, g(\mathbf{x})) = 0.$$

Moreover, if $\mathbf{x}$ in U and z in V satisfy $F(\mathbf{x}, z) = 0$, then $z = g(\mathbf{x})$. Finally, $z = g(\mathbf{x})$ is continuously differentiable, with the derivative given by

$$\mathbf{D}g(\mathbf{x}) = -\frac{1}{\frac{\partial F}{\partial z}(\mathbf{x}, z)} \mathbf{D}_{\mathbf{x}} F(\mathbf{x}, z)\Big|_{z=g(\mathbf{x})},$$

where $\mathbf{D}_{\mathbf{x}} F$ denotes the (partial) derivative of F with respect to the variable $\mathbf{x}$—that is, we have $\mathbf{D}_{\mathbf{x}} F = [\partial F/\partial x_1, \ldots, \partial F/\partial x_n]$; in other words,

$$\frac{\partial g}{\partial x_i} = -\frac{\partial F/\partial x_i}{\partial F/\partial z}, \qquad i = 1, \ldots, n. \tag{1}$$

Implicit Differentiation

The Chain Rule can be used to give a more complete description of the process of implicit differentiation.

We suppose that an equation of the form $F(x, y) = 0$ defines y implicitly as a differentiable function of x, that is, $y = f(x)$, where $F(x, f(x)) = 0$ for all x in the domain of f.

If F is differentiable, we can apply Case 1 of the Chain Rule to differentiate both sides of the equation $F(x, y) = 0$ with respect to x.

Since both x and y are functions of x, we obtain

$$\frac{\partial F}{\partial x}\frac{dx}{dx} + \frac{\partial F}{\partial y}\frac{dy}{dx} = 0$$

Implicit Differentiation

But $dx/dx = 1$, so if $\partial F/\partial x \neq 0$ we solve for dy/dx and obtain

6
$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

To derive this equation we assumed that $F(x, y) = 0$ defines y implicitly as a function of x.

Implicit Differentiation

The **Implicit Function Theorem**, proved in advanced calculus, gives conditions under which this assumption is valid:

It states that if F is defined on a disk containing (a, b), where $F(a, b) = 0$, $F_y(a, b) \neq 0$, and F_x and F_y are continuous on the disk, then the equation $F(x, y) = 0$ defines y as a function of x near the point (a, b) and the derivative of this function is given by Equation 6.

Example 8

Find y' if $x^3 + y^3 = 6xy$.

Solution:
The given equation can be written as

$$F(x, y) = x^3 + y^3 - 6xy = 0$$

so Equation 6 gives

$$\frac{dy}{dx} = -\frac{F_x}{F_y}$$

$$= -\frac{3x^2 - 6y}{3y^2 - 6x} = -\frac{x^2 - 2y}{y^2 - 2x}$$

Implicit Differentiation

Now we suppose that z is given implicitly as a function $z = f(x, y)$ by an equation of the form $F(x, y, z) = 0$.

This means that $F(x, y, f(x, y)) = 0$ for all (x, y) in the domain of f. If F and f are differentiable, then we can use the Chain Rule to differentiate the equation $F(x, y, z) = 0$ as follows:

$$\frac{\partial F}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

Implicit Differentiation

But $\frac{\partial}{\partial x}(x) = 1$ and $\frac{\partial}{\partial x}(y) = 0$

so this equation becomes

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0$$

If $\partial F/\partial z \neq 0$, we solve for $\partial z/\partial x$ and obtain the first formula in Equations 7.

The formula for $\partial z/\partial y$ is obtained in a similar manner.

Implicit Differentiation

7 $$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Again, a version of the **Implicit Function Theorem** stipulates conditions under which our assumption is valid:

If F is defined within a sphere containing (a, b, c), where $F(a, b, c) = 0$, $F_z(a, b, c) \neq 0$, and F_x, F_y, and F_z are continuous inside the sphere, then the equation $F(x, y, z) = 0$ defines z as a function of x and y near the point (a, b, c) and this function is differentiable, with partial derivatives given by 7.