Lagrange Multipliers

In this section we present Lagrange's method for maximizing or minimizing a general function $f(x, y, z)$ subject to a constraint (or side condition) of the form $g(x, y, z) = k$.

It's easier to explain the geometric basis of Lagrange's method for functions of two variables. So we start by trying to find the extreme values of $f(x, y)$ subject to a constraint of the form $g(x, y) = k$.

In other words, we seek the extreme values of $f(x, y)$ when the point (x, y) is restricted to lie on the level curve $g(x, y) = k$.

Lagrange Multipliers

Figure 1 shows this curve together with several level curves of f.

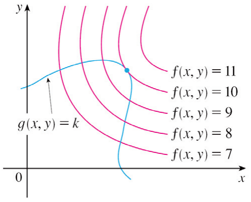

Figure 1

These have the equations $f(x, y) = c$, where $c = 7, 8, 9, 10, 11$.

Lagrange Multipliers

To maximize $f(x, y)$ subject to $g(x, y) = k$ is to find the largest value of c such that the level curve $f(x, y) = c$ intersects $g(x, y) = k$.

It appears from Figure 1 that this happens when these curves just touch each other, that is, when they have a common tangent line. (Otherwise, the value of c could be increased further.)

Lagrange Multipliers

This means that the normal lines at the point (x_0, y_0) where they touch are identical. So the gradient vectors are parallel; that is, $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$ for some scalar λ.

This kind of argument also applies to the problem of finding the extreme values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$.

Thus the point (x, y, z) is restricted to lie on the level surface S with equation $g(x, y, z) = k$.

Lagrange Multipliers

Instead of the level curves in Figure 1, we consider the level surfaces $f(x, y, z) = c$ and argue that if the maximum value of f is $f(x_0, y_0, z_0) = c$, then the level surface $f(x, y, z) = c$ is tangent to the level surface $g(x, y, z) = k$ and so the corresponding gradient vectors are parallel.

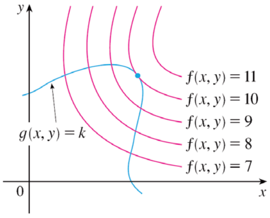

Figure 1

Lagrange Multipliers

This intuitive argument can be made precise as follows. Suppose that a function f has an extreme value at a point $P(x_0, y_0, z_0)$ on the surface S and let C be a curve with vector equation $\mathbf{r}(t) = \langle x(t), y(t), z(t)\rangle$ that lies on S and passes through P.

If t_0 is the parameter value corresponding to the point P, then $\mathbf{r}(t_0) = \langle x_0, y_0, z_0\rangle$.

The composite function $h(t) = f(x(t), y(t), z(t))$ represents the values that f takes on the curve C.

Lagrange Multipliers

Since f has an extreme value at (x_0, y_0, z_0), it follows that h has an extreme value at t_0, so $h'(t_0) = 0$. But if f is differentiable, we can use the Chain Rule to write

$$\begin{aligned} 0 &= h'(t_0) \\ &= f_x(x_0, y_0, z_0)x'(t_0) + f_y(x_0, y_0, z_0)y'(t_0) + f_z(x_0, y_0, z_0)z'(t_0) \\ &= \nabla f(x_0, y_0, z_0) \cdot \mathbf{r}'(t_0) \end{aligned}$$

This shows that the gradient vector $\nabla f(x_0, y_0, z_0)$ is orthogonal to the tangent vector $\mathbf{r}'(t_0)$ to every such curve C. But we already know that the gradient vector of g, $\nabla g(x_0, y_0, z_0)$, is also orthogonal to $\mathbf{r}'(t_0)$ for every such curve.

Lagrange Multipliers

This means that the gradient vectors $\nabla f(x_0, y_0, z_0)$ and $\nabla g(x_0, y_0, z_0)$ must be parallel. Therefore, if $\nabla g(x_0, y_0, z_0) \neq \mathbf{0}$, there is a number λ such that

$$\boxed{1} \qquad \boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)}$$

The number λ in Equation 1 is called a **Lagrange multiplier**.

Lagrange Multipliers

The procedure based on Equation 1 is as follows.

Method of Lagrange Multipliers To find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = k$ [assuming that these extreme values exist and $\nabla g \neq \mathbf{0}$ on the surface $g(x, y, z) = k$]:

(a) Find all values of x, y, z, and λ such that

$$\nabla f(x, y, z) = \lambda \nabla g(x, y, z)$$

and

$$g(x, y, z) = k$$

(b) Evaluate f at all the points (x, y, z) that result from step (a). The largest of these values is the maximum value of f; the smallest is the minimum value of f.

Lagrange Multipliers

If we write the vector equation $\nabla f = \lambda \nabla g$ in terms of components, then the equations in step (a) become

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad f_z = \lambda g_z \qquad g(x, y, z) = k$$

This is a system of four equations in the four unknowns x, y, z, and λ, but it is not necessary to find explicit values for λ.

For functions of two variables the method of Lagrange multipliers is similar to the method just described.

Lagrange Multipliers

To find the extreme values of $f(x, y)$ subject to the constraint $g(x, y) = k$, we look for values of x, y, and λ such that

$$\nabla f(x, y) = \lambda \nabla g(x, y) \quad \text{and} \quad g(x, y) = k$$

This amounts to solving three equations in three unknowns:

$$f_x = \lambda g_x \qquad f_y = \lambda g_y \qquad g(x, y) = k$$

Example 1 – *Maximizing a volume using Lagrange multipliers*

A rectangular box without a lid is to be made from 12 m^2 of cardboard. Find the maximum volume of such a box.

Solution:

Let x, y, and z be the length, width, and height, respectively, of the box in meters.

Then we wish to maximize

$$V = xyz$$

subject to the constraint

$$g(x, y, z) = 2xz + 2yz + xy = 12$$

Example 1 – *Solution*

Using the method of Lagrange multipliers, we look for values of x, y, z, and λ such that $\nabla V = \lambda \nabla g$ and $g(x, y, z) = 12$.

This gives the equations

$$V_x = \lambda g_x$$

$$V_y = \lambda g_y$$

$$V_z = \lambda g_z$$

$$2xz + 2yz + xy = 12$$

Example 1 – *Solution*

Which become

$\boxed{2}$ $\quad yz = \lambda(2z + y)$

$\boxed{3}$ $\quad xz = \lambda(2z + x)$

$\boxed{4}$ $\quad xy = \lambda(2x + 2y)$

$\boxed{5}$ $\quad 2xz + 2yz + xy = 12$

Example 1 – *Solution*

There are no general rules for solving systems of equations. Sometimes some ingenuity is required.

In the present example you might notice that if we multiply $\boxed{2}$ by x, $\boxed{3}$ by y, and $\boxed{4}$ by z, then the left sides of these equations will be identical.

Doing this, we have

$\boxed{6}$ $\quad xyz = \lambda(2xz + xy)$

$\boxed{7}$ $\quad xyz = \lambda(2yz + xy)$

$\boxed{8}$ $\quad xyz = \lambda(2xz + 2yz)$

Example 1 – *Solution*

We observe that $\lambda \neq 0$ because $\lambda = 0$ would imply $yz = xz = xy = 0$ from $\boxed{2}$, $\boxed{3}$, and $\boxed{4}$ and this would contradict $\boxed{5}$.

Therefore, from $\boxed{6}$ and $\boxed{7}$, we have

$$2xz + xy = 2yz + xy$$

which gives $xz = yz$.

But $z \neq 0$ (since $z = 0$ would give $V = 0$), so $x = y$.

Example 1 – *Solution*

From $\boxed{7}$ and $\boxed{8}$ we have

$$2yz + xy = 2xz + 2yz$$

which gives $2xz = xy$ and so (since $x \neq 0$) $y = 2z$.

If we now put $x = y = 2z$ in $\boxed{5}$, we get

$$4z^2 + 4z^2 + 4z^2 = 12$$

Since x, y, and z are all positive, we therefore have $z = 1$ and so $x = 2$ and $y = 2$.

Two Constraints

Two Constraints

Suppose now that we want to find the maximum and minimum values of a function $f(x, y, z)$ subject to two constraints (side conditions) of the form $g(x, y, z) = k$ and $h(x, y, z) = c$.

Geometrically, this means that we are looking for the extreme values of f when (x, y, z) is restricted to lie on the curve of intersection C of the level surfaces $g(x, y, z) = k$ and $h(x, y, z) = c$. (See Figure 5.)

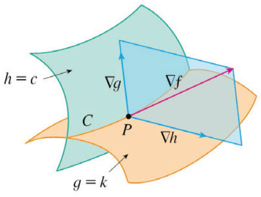

Figure 5

Two Constraints

Suppose f has such an extreme value at a point $P(x_0, y_0, z_0)$. We know from the beginning of this section that ∇f is orthogonal to C at P.

But we also know that ∇g is orthogonal to $g(x, y, z) = k$ and ∇h is orthogonal to $h(x, y, z) = c$, so ∇g and ∇h are both orthogonal to C.

This means that the gradient vector $\nabla f(x_0, y_0, z_0)$ is in the plane determined by $\nabla g(x_0, y_0, z_0)$ and $\nabla h(x_0, y_0, z_0)$. (We assume that these gradient vectors are not zero and not parallel.)

Two Constraints

So there are numbers λ and μ (called Lagrange multipliers) such that

$$\boxed{\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0) + \mu \nabla h(x_0, y_0, z_0)} \tag{16}$$

In this case Lagrange's method is to look for extreme values by solving five equations in the five unknowns x, y, z, λ, and μ.

Two Constraints

These equations are obtained by writing Equation 16 in terms of its components and using the constraint equations:

$$f_x = \lambda g_x + \mu h_x$$

$$f_y = \lambda g_y + \mu h_y$$

$$f_z = \lambda g_z + \mu h_z$$

$$g(x, y, z) = k$$

$$h(x, y, z) = c$$

Example 5 – *A maximum problem with two constraints*

Find the maximum value of the function $f(x, y, z) = x + 2y + 3z$ on the curve of intersection of the plane $x - y + z = 1$ and the cylinder $x^2 + y^2 = 1$.

Solution:

We maximize the function $f(x, y, z) = x + 2y + 3z$ subject to the constraints $g(x, y, z) = x - y + z = 1$ and $h(x, y, z) = x^2 + y^2 = 1$.

Example 5 – *Solution*

cont'd

The Lagrange condition is $\nabla f = \lambda \nabla g + \mu \nabla h$, so we solve the equations

$$1 = \lambda + 2x\mu \tag{17}$$

$$2 = -\lambda + 2y\mu \tag{18}$$

$$3 = \lambda \tag{19}$$

$$x - y + z = 1 \tag{20}$$

$$x^2 + y^2 = 1 \tag{21}$$

Putting $\lambda = 3$ [from $\boxed{19}$] in $\boxed{17}$, we get $2x\mu = -2$, so $x = -1/\mu$. Similarly, $\boxed{18}$ gives $y = 5/(2\mu)$.

Example 5 – *Solution*

cont'd

Substitution in $\boxed{21}$ then gives

$$\frac{1}{\mu^2} + \frac{25}{4\mu^2} = 1$$

and so $\mu^2 = \frac{29}{4}$, $\mu = \pm\sqrt{29}/2$.

Then $x = \mp 2/\sqrt{29}$, $y = \pm 5/\sqrt{29}$, and, from $\boxed{20}$, $z = 1 - x + y = 1 \pm 7/\sqrt{29}$.

Example 5 – *Solution*

cont'd

The corresponding values of *f* are

$$\mp \frac{2}{\sqrt{29}} + 2\left(\pm \frac{5}{\sqrt{29}}\right) + 3\left(1 \pm \frac{7}{\sqrt{29}}\right) = 3 \pm \sqrt{29}$$

Therefore the maximum value of *f* on the given curve is $3 + \sqrt{29}$.

3.4 Constrained Extrema and Lagrange Multipliers

Key Points in this Section.

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1. **Lagrange Multiplier Equations.** Let $f : U \subset \mathbb{R}^n \to \mathbb{R}$ and $g : U \subset \mathbb{R}^n \to \mathbb{R}$ be C^1. Consider the problem of extremizing f on a level set of g, say $g(\mathbf{x}) = c$. If $\mathbf{x}_0$ is such an extremum and if $\nabla g(\mathbf{x}_0) \neq \mathbf{0}$ then the ***Lagrange multiplier equations*** hold:

$$\nabla f(\mathbf{x}_0) = \lambda \nabla g(\mathbf{x}_0)$$

   for a constant λ, the ***multiplier***.

2. The idea of the proof is to use the fact that f has a critical point along any curve in the level set through $\mathbf{x}_0$, which shows, via the chain rule, that $\nabla f(\mathbf{x}_0)$ is perpendicular to that level set; but $\nabla g(\mathbf{x}_0)$ is also perpendicular, so these two vectors are parallel.

3. The Lagrange multiplier method produces *candidates* for extrema; one must make sure there is an extremum and then f can be evaluated at the candidates to choose the maximum or minimum as desired.

4. If there are k constraints

$$g_1 = c_1, \cdots, g_k = c_k,$$

for C^1 functions $g(x_1, \ldots, x_n), \ldots g_k(x_1, \ldots, x_n)$ and constants $c_1, \ldots, c_k$, then the Lagrange multiplier equations become

$$\nabla f(\mathbf{x}_0) = \lambda_1 \nabla g(\mathbf{x}_0) + \cdots + \lambda_k \nabla g(\mathbf{x}_0).$$

5. The Lagrange multiplier method is an effective tool for finding the extrema of $f|\partial U$ in the strategy for finding global extrema described in the last section.

6. **Second Derivative Test with Constraints.** Let $\mathbf{x}_0$ satisfy the conditions of the Lagrange multiplier theorem (in point 1.) Let $h = f - \lambda g$ and $|\bar{H}|$ be the ***bordered Hessian determinant***:

$$|\bar{H}| = \begin{vmatrix} 0 & -\dfrac{\partial g}{\partial x} & -\dfrac{\partial g}{\partial y} \\ -\dfrac{\partial g}{\partial x} & \dfrac{\partial^2 h}{\partial x^2} & \dfrac{\partial^2 h}{\partial x \partial y} \\ -\dfrac{\partial g}{\partial y} & \dfrac{\partial^2 h}{\partial x \partial y} & \dfrac{\partial^2 h}{\partial y^2} \end{vmatrix}$$

evaluated at $\mathbf{x}_0$.

If $|\bar{H}| > 0$, then $\mathbf{x}_0$ is a local maximum of f subject to the constraint $g = c$ and if $|\bar{H}| < 0$, it is a local minimum.