3.2 Taylor's Theorem

Key Points in this Section.

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1. The one-variable ***Taylor Theorem*** states that if f is C^{k+1}, then

$$f(x_0+h) = f(x_0)+f'(x_0)h+\frac{f''(x_0)}{2}h^2+\cdots+\frac{f^{(k)}(x_0)}{k!}h^k+R_k(x_0,h),$$

where $R_k(x_0,h)/h^k \to 0$ as $h \to 0$

2. The idea of the proof is to start with the Fundamental Theorem of Calculus

$$f(x_0+h) = f(x_0) + \int_{x_0}^{x_0+h} f'(\tau)d\tau$$

(which gives Taylor's theorem for $k=0$) and integrating by parts.

3. For $f : U \subset \mathbb{R}^n \to \mathbb{R}$ of class C^3, the second-order ***Taylor Theorem*** states that

$$f(\mathbf{x}_0+\mathbf{h}) = f(\mathbf{x}_0)+\sum_{i=1}^{n} h_i\frac{\partial f}{\partial x_i}(\mathbf{x}_0)+\frac{1}{2}\sum_{i,j} h_ih_j\frac{\partial^2 f}{\partial x_i\partial x_j}(\mathbf{x}_0)+R_2(\mathbf{x}_0,\mathbf{h})$$

where $R_2(\mathbf{x}_0, \mathbf{h})/\|\mathbf{h}\|^2 \to 0$ as $\mathbf{h} \to \mathbf{0}$. Higher order versions are similar.

4. The idea of the proof is to apply the single-variable Taylor theorem to the function $g(t) = f(\mathbf{x}_0 + t\mathbf{h})$, expanded about $t_0 = 0$ with $h = 1$.
