### 3.1 Iterated Partial Derivatives

**Key Points in this Section.**

1. **Equality of Mixed Partials.** If $f(x, y)$ is $C^2$ (has continuous 2nd partial derivatives), then
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}.$$

2. The idea of the proof is to apply the mean value theorem to the "difference of differences" written in the two ways
$$\begin{aligned} S(h,k) &= \{f(x+h, y+k) - f(x+h, y)\} - \{f(x, y+k) - f(x, y)\} \\ &= \{f(x+h, y+k) - f(x, y+k)\} - \{f(x+h, y) - f(x, y)\} \end{aligned}$$

3. Higher order partials are also symmetric; for example, for $f(x, y, z)$,
$$\frac{\partial^4 f}{\partial x \partial^2 z \partial y} = \frac{\partial^4 f}{\partial x \partial y \partial^2 z}$$

4. Many important equations describing nature involve partial derivatives, such as the ***heat equation*** for the temperature $T(x, y, z, t)$:
$$\frac{\partial T}{\partial t} = k\left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2}\right).$$

## Higher Derivatives



## Higher Derivatives

If $f$ is a function of two variables, then its partial derivatives $f_x$ and $f_y$ are also functions of two variables, so we can consider their partial derivatives $(f_x)_x$, $(f_x)_y$, $(f_y)_x$, and $(f_y)_y$, which are called the **second partial derivatives** of $f$.

If $z = f(x, y)$, we use the following notation:

$$(f_x)_x = f_{xx} = f_{11} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial^2 z}{\partial x^2}$$

$$(f_x)_y = f_{xy} = f_{12} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y\, \partial x} = \frac{\partial^2 z}{\partial y\, \partial x}$$



## Higher Derivatives

$$(f_y)_x = f_{yx} = f_{21} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x\, \partial y} = \frac{\partial^2 z}{\partial x\, \partial y}$$

$$(f_y)_y = f_{yy} = f_{22} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial^2 z}{\partial y^2}$$

Thus the notation $f_{xy}$ (or $\partial^2 f/\partial y\, \partial x$) means that we first differentiate with respect to $x$ and then with respect to $y$, whereas in computing $f_{yx}$ the order is reversed.



## Example 6

Find the second partial derivatives of

$$f(x, y) = x^3 + x^2y^3 - 2y^2$$

Solution:

In Example 1 we found that

$f_x(x, y) = 3x^2 + 2xy^3 \qquad f_y(x, y) = 3x^2y^2 - 4y$

Therefore

$$f_{xx} = \frac{\partial}{\partial x}(3x^2 + 2xy^3)$$
$$= 6x + 2y^3$$



## Example 6 – *Solution*

cont'd

$$f_{xy} = \frac{\partial}{\partial y}(3x^2 + 2xy^3)$$
$$= 6xy^2$$

$$f_{yx} = \frac{\partial}{\partial x}(3x^2y^2 - 4y)$$
$$= 6xy^2$$

$$f_{yy} = \frac{\partial}{\partial y}(3x^2y^2 - 4y)$$
$$= 6x^2y - 4$$

# Higher Derivatives

Notice that $f_{xy} = f_{yx}$ in Example 6. This is not just a coincidence.

It turns out that the mixed partial derivatives $f_{xy}$ and $f_{yx}$ are equal for most functions that one meets in practice.

The following theorem, which was discovered by the French mathematician Alexis Clairaut (1713–1765), gives conditions under which we can assert that $f_{xy} = f_{yx}$.

**Clairaut's Theorem** Suppose $f$ is defined on a disk $D$ that contains the point $(a, b)$. If the functions $f_{xy}$ and $f_{yx}$ are both continuous on $D$, then

$$f_{xy}(a, b) = f_{yx}(a, b)$$



# Higher Derivatives

Partial derivatives of order 3 or higher can also be defined. For instance,

$$f_{xyy} = (f_{xy})_y = \frac{\partial}{\partial y}\left(\frac{\partial^2 f}{\partial y\, \partial x}\right) = \frac{\partial^3 f}{\partial y^2\, \partial x}$$

and using Clairaut's Theorem it can be shown that $f_{xyy} = f_{yxy} = f_{yyx}$ if these functions are continuous.