NOTATION:

$$R^n = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} : x_1, \ldots, x_n \in R \right\}$$

EXAMPLE:
Let

$$\bar{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ \bar{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \bar{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is a basis for R^n.

DEFINITION:
The set $\{\bar{e}_1, \ldots, \bar{e}_n\}$ is called the standard basis for R^n.

DEFINITION:

An $m \times n$ matrix is an array of mn numbers:

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

DEFINITION:

If A and B are $m \times n$ matrices, then the sum $A+B$ is the $m \times n$ matrix whose entries are the sums of the corresponding entries of A and B.

EXAMPLE:

$$\begin{bmatrix} 1 & -2 & -1 \\ -2 & 3 & -2 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 2 \\ -2 & -4 & -1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ -4 & -1 & -3 \end{bmatrix}$$

REMARK: We can add matrices only of the same size.

DEFINITION:

If r is a scalar and A is a matrix, then the scalar multiple rA is the matrix whose entries are r times the corresponding entries in A.

EXAMPLE:

$$(-2) \begin{bmatrix} 1 & 2 & -3 \\ -1 & 0 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 & 6 \\ 2 & 0 & 4 \end{bmatrix}$$

PROPERTIES:

Let A, B, and C be matrices of the same size, and let r and s be scalars. Then

(a) $A + B = B + A$

(b) $(A + B) + C = A + (B + C)$

(c) $r(A + B) = rA + rB$

(d) $(r + s)A = rA + sA$

(e) $r(sA) = (rs)A$

DEFINITION:

If A is an $m \times n$ matrix, and if B is an $n \times p$ matrix with columns $\bar{b}_1, \ldots, \bar{b}_p$, then the product AB is the $m \times p$ matrix whose columns are $A\bar{b}_1, \ldots, A\bar{b}_p$. That is,

$$AB = A[\bar{b}_1 \ \ \bar{b}_2 \ \ldots \ \bar{b}_p] = [A\bar{b}_1 \ \ A\bar{b}_2 \ \ldots \ A\bar{b}_p]$$

ROW-COLUMN RULE FOR COMPUTING AB:

If the product AB is defined, then the entry in row i and column j of AB is the sum of the products of corresponding entries from row i of A and column j of B. If $(AB)_{ij}$ denotes the (i,j)-entry in AB, and if A is an $m \times n$ matrix, then

$$(AB)_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \ldots + a_{in}b_{nj}$$

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 3 \\ 1 & -5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 3 & 6 \\ 1 & -2 & 3 \end{bmatrix}$, then

AB

$$= \begin{bmatrix} 2\cdot 4 + 3\cdot 1 & 2\cdot 3 + 3\cdot(-2) & 2\cdot 6 + 3\cdot 3 \\ 1\cdot 4 + (-5)\cdot 1 & 1\cdot 3 + (-5)\cdot(-2) & 1\cdot 6 + (-5)\cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 0 & 21 \\ -1 & 13 & -9 \end{bmatrix}$$

Note that BA is undefined.

PROBLEM:

Consider the following matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix}, C = \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}.$$

If possible, compute:

(a) AB

(b) $AC + B^2$

(c) $AB + C^2$

SOLUTION:

We have:

(a) $AB = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix}$.

(b) Impossible.

(c) $AB + C^2$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & -2 \\ 2 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & 8 \\ 16 & 9 \end{bmatrix} + \begin{bmatrix} 5 & -16 \\ 16 & 21 \end{bmatrix}$$

$$= \begin{bmatrix} 19 & -8 \\ 32 & 30 \end{bmatrix}.$$

PROPERTIES:

Let A be an $m \times n$ matrix, and let B and C have sizes for which the indicated sums and products are defined. Then

(a) $A(BC) = (AB)C$

(b) $A(B + C) = AB + AC$

(c) $(B + C)A = BA + CA$

(d) $r(AB) = (rA)B = A(rB)$

WARNING

1. In general, $AB \neq BA$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$ **and** $B = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix}$. **Then**

$$AB = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 1 \\ 2 & 0 \end{bmatrix}$$

$$BA = \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

So,

$$AB \neq BA.$$

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$ **and** $B = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$, **then**

$$AB = \begin{bmatrix} 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \end{bmatrix} = [32]$$

and

$$BA = \begin{bmatrix} 4 \cdot 1 & 4 \cdot 2 & 4 \cdot 3 \\ 5 \cdot 1 & 5 \cdot 2 & 5 \cdot 3 \\ 6 \cdot 1 & 6 \cdot 2 & 6 \cdot 3 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 8 & 12 \\ 5 & 10 & 15 \\ 6 & 12 & 18 \end{bmatrix}$$

WARNING

2. If $AB = AC$, then it is not true in general that $B = C$.

EXAMPLE: Let

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \text{and } C = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}.$$

Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = AC, \quad \text{but} \quad B \neq C.$$

WARNING

3. If $AB = 0$, then it is not true in general that $A = 0$ or $B = 0$.

EXAMPLE:

Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

So,

$$AB = 0, \quad \text{but} \quad A \neq 0 \quad \text{and} \quad B \neq 0.$$

THE TRANSPOSE OF A MATRIX

DEFINITION:

Let A be an $m \times n$ matrix. The transpose of A is the $n \times m$ matrix, denoted by A^T, whose columns are formed from the corresponding rows of A.

EXAMPLE:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} -3 & 1 \\ 4 & 7 \\ 8 & -5 \end{bmatrix} \qquad B^T = \begin{bmatrix} -3 & 4 & 8 \\ 1 & 7 & -5 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 5 \\ 2 & 6 \\ 3 & 7 \\ 4 & 8 \end{bmatrix}$$

PROPERTIES:

Let A and B denote matrices whose sizes are appropriate for the following sums and products. Then

(a) $(A^T)^T = A$

(b) $(A + B)^T = A^T + B^T$

(c) $(rA)^T = rA^T$ for any scalar r

(d) $(AB)^T = B^T A^T$

THE INVERSE OF A MATRIX

DEFINITION:

The identity matrix I is the $n \times n$ matrix of the form

$$I = \begin{bmatrix} 1 & 0 & 0 & \ldots & 0 & 0 \\ 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \ldots & & & & & \ldots \\ 0 & 0 & 0 & \ldots & 1 & 0 \\ 0 & 0 & 0 & \ldots & 0 & 1 \end{bmatrix}$$

MAIN PROPERTY:

$$AI = IA = A$$

DEFINITION:

An $n \times n$ matrix A is said to be invertible if there is an $n \times n$ matrix C such that

$$CA = I \quad \text{and} \quad AC = I.$$

In this case, C is an inverse of A and is denoted by A^{-1}. So,

$$A^{-1}A = I \quad \text{and} \quad AA^{-1} = I.$$

PROPERTIES:

Let A and B be invertible $n \times n$ matrices. Then

(a) $(A^{-1})^{-1} = A$

(b) $(AB)^{-1} = B^{-1}A^{-1}$

(c) $(A^T)^{-1} = (A^{-1})^T$

THEOREM:

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad - bc \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad - bc = 0$, then A is not invertible.

EXAMPLE:

Let $A = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix}$. Then $A^{-1} = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix}$. In fact, we have

$$AA^{-1} = \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$A^{-1}A = \begin{bmatrix} -7 & -5 \\ 3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -3 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

ALGORITHM FOR FINDING A^{-1}:

1. Row reduce the augmented matrix $[A \ I]$.

2. If A is row equivalent to I, then $[A \ I]$ is row equivalent to $[I \ A^{-1}]$.

3. Otherwise, A does not have an inverse.

Let

$$A = \begin{bmatrix} 3 & -1 & 1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

We find A^{-1}:

$$\left[\begin{array}{rrr|rrr} 3 & -1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ -1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 2 & 1 & 1 & 3 & 0 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & -2 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 1 & 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{rrr|rrr} 1 & 0 & 0 & 1 & 1 & -1 \\ 0 & 1 & 0 & 1 & 2 & -1 \\ 0 & 0 & 1 & -1 & -1 & 2 \end{array}\right]$$

therefore

$$A^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$$

DEFINITION:

The determinant of an $n \times n$ matrix A is the following sum:

$$\det A = a_{11} \det A_{11} - a_{12} \det A_{12} + a_{13} \det A_{13} - \dots + (-1)^{n+1} a_{1n} \det A_{1n},$$

where A_{1j} are submatrices formed by deleting from A the first row and jth column.

PROBLEM: Find $\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix}$.

SOLUTION: We have

$$\begin{vmatrix} 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1 \\ -1 & 0 & 3 & 0 \\ 0 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} - \begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

Since

$$\begin{vmatrix} 2 & 0 & -1 \\ 0 & 3 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 2 \begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 3 \\ 1 & 1 \end{vmatrix} = 2(3-0) + (-1)(0-3) = 9$$

and

$$\begin{vmatrix} 0 & 2 & 0 \\ -1 & 0 & 3 \\ 0 & 1 & 1 \end{vmatrix} = -2 \begin{vmatrix} -1 & 3 \\ 0 & 1 \end{vmatrix} = -2(-1) = 2$$

it follows that the determinant is equal to $9 - 2 = 7$.

THEOREM:

If A is a triangular matrix, then $\det A$ is the product of the entries on the main diagonal of A.

EXAMPLE:

$$\begin{vmatrix} 3 & -7 & 8 & 9 & -6 \\ 0 & 2 & -5 & 7 & 3 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 4 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix} = 3 \cdot 2 \cdot 1 \cdot 4 \cdot 1 = 24$$

THEOREM:

We have $\det A = 0$

(a) if A contains a zero-row or zero-column.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{vmatrix} = 0$

(b) if A contains two similar rows or columns.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ 2 & 5 & 2 \\ 2 & 0 & 2 \end{vmatrix} = 0$

(c) if some row (column) of A is a multiple of some other row (column) of A.

EXAMPLE: $\begin{vmatrix} 1 & 2 & 1 \\ -2 & -4 & -2 \\ 3 & 5 & 7 \end{vmatrix} = 0$

THEOREM:

Let A be a square matrix.

(a) If a multiple of one row (column) of A is added to another row (column) to produce a matrix B, then $\det A = \det B$.

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{vmatrix} = 2.$

(b) If two rows (columns) of A are interchanged to produce B, then $\det A = -\det B$.

EXAMPLE: $\begin{vmatrix} 1 & 1 & 2 \\ 1 & 2 & 3 \\ 3 & 3 & 8 \end{vmatrix} = -\begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 3 & 3 & 8 \end{vmatrix}$

(c) If one row (column) of A is multiplied by k to produce B, then $\det B = k \det A$.

EXAMPLE: $\begin{vmatrix} 100 & 300 \\ 1 & 2 \end{vmatrix} = 100 \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix}.$

PROBLEM: Find

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix}$$

SOLUTION: We have

$$\begin{vmatrix} 1 & 3 & 5 & 4 \\ 2 & -3 & 1 & -1 \\ -1 & 2 & -1 & 0 \\ 2 & 2 & 5 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & -9 & -9 & -9 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9)\begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 5 & 4 & 4 \\ 0 & -4 & -5 & -5 \end{vmatrix}$$

$$= (-9)\begin{vmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -1 & -1 \\ 0 & 0 & -1 & -1 \end{vmatrix}$$

Since the last two rows are equal, the determinant is equal to 0.

THEOREM:

A square matrix A is invertible if and only if $\det A \neq 0$.

THEOREM:

Let A be a square matrix. Then

(a) $\det A^T = \det A$.

(b) $\det(AB) = \det A \det B$.