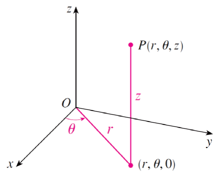
Cylindrical Coordinates

Cylindrical Coordinates

In the **cylindrical coordinate system**, a point *P* in three-dimensional space is represented by the ordered triple (r, θ, z) where r and θ are polar coordinates of the projection of *P* onto the *xy*-plane and *z* is the directed distance from the *xy*-plane to *P*. (See Figure 2.)



The cylindrical coordinates of a point

Figure 2

Cylindrical Coordinates

To convert from cylindrical to rectangular coordinates, we use the equations

1 $$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

whereas to convert from rectangular to cylindrical coordinates, we use

2 $$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x} \qquad z = z$$

Example 1

(a) Plot the point with cylindrical coordinates (2, $2\pi/3$, 1) and find its rectangular coordinates.

(b) Find cylindrical coordinates of the point with rectangular coordinates (3, –3, –7).

Solution:

(a) The point with cylindrical coordinates (2, $2\pi/3$, 1) is plotted in Figure 3.

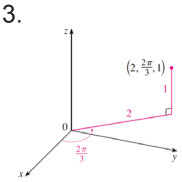


Figure 3

Example 1 – *Solution*

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From Equations 1, its rectangular coordinates are

$$x = 2\cos\frac{2\pi}{3} = 2\left(-\frac{1}{2}\right) = -1$$

$$y = 2\sin\frac{2\pi}{3} = 2\left(\frac{\sqrt{3}}{2}\right) = \sqrt{3}$$

$$z = 1$$

Thus the point is (-1, $\sqrt{3}$, 1) in rectangular coordinates.

Example 1 – *Solution*

cont'd

(b) From Equations 2 we have

$$r = \sqrt{3^2 + (-3)^2} = 3\sqrt{2}$$

$$\tan\theta = \frac{-3}{3} = -1 \qquad \text{so} \qquad \theta = \frac{7\pi}{4} + 2n\pi$$

$$z = -7$$

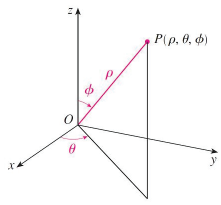
Therefore one set of cylindrical coordinates is ($3\sqrt{2}$, $7\pi/4$, –7). Another is ($3\sqrt{2}$, $-\pi/4$, –7).

As with polar coordinates, there are infinitely many choices.

Spherical Coordinates

Spherical Coordinates

The **spherical coordinates** (ρ, θ, ϕ) of a point P in space are shown in Figure 1, where $\rho = |OP|$ is the distance from the origin to P, θ is the same angle as in cylindrical coordinates, and ϕ is the angle between the positive z-axis and the line segment OP.



The spherical coordinates of a point

Figure 1

Spherical Coordinates

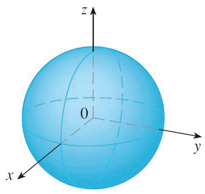
Note that

$$\rho \geq 0 \qquad\qquad 0 \leq \phi \leq \pi$$

The spherical coordinate system is especially useful in problems where there is symmetry about a point, and the origin is placed at this point.

Spherical Coordinates

For example, the sphere with center the origin and radius c has the simple equation $\rho = c$ (see Figure 2); this is the reason for the name "spherical" coordinates.

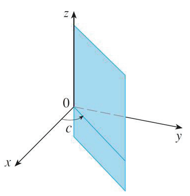


$\rho = c$, a sphere

Figure 2

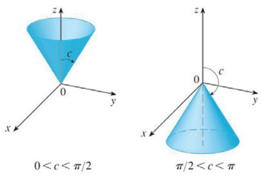
Spherical Coordinates

The graph of the equation $\theta = c$ is a vertical half-plane (see Figure 3), and the equation $\phi = c$ represents a half-cone with the z-axis as its axis (see Figure 4).



$\theta = c$, a half-plane

Figure 3



$\phi = c$, a half-plane

Figure 4

Spherical Coordinates

The relationship between rectangular and spherical coordinates can be seen from Figure 5.

From triangles OPQ and OPP' we have

$$z = \rho \cos \phi \qquad\qquad r = \rho \sin \phi$$

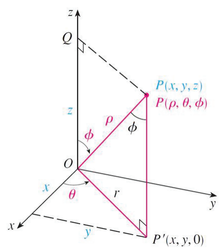


Figure 5

Spherical Coordinates

But $x = r \cos\theta$ and $y = r \sin\theta$, so to convert from spherical to rectangular coordinates, we use the equations

1 $$\boxed{x = \rho \sin\phi \cos\theta \qquad y = \rho \sin\phi \sin\theta \qquad z = \rho \cos\phi}$$

Also, the distance formula shows that

2 $$\boxed{\rho^2 = x^2 + y^2 + z^2}$$

We use this equation in converting from rectangular to spherical coordinates.

Example 1

The point $(2, \pi/4, \pi/3)$ is given in spherical coordinates. Plot the point and find its rectangular coordinates.

Solution:

We plot the point in Figure 6.

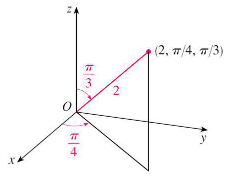


Figure 6

Example 1 – *Solution*

cont'd

From Equations 1 we have

$$x = \rho \sin\phi \cos\theta = 2 \sin\frac{\pi}{3} \cos\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$y = \rho \sin\phi \sin\theta = 2 \sin\frac{\pi}{3} \sin\frac{\pi}{4} = 2\left(\frac{\sqrt{3}}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \sqrt{\frac{3}{2}}$$

$$z = \rho \cos\phi = 2 \cos\frac{\pi}{3} = 2\left(\tfrac{1}{2}\right) = 1$$

Thus the point $(2, \pi/4, \pi/3)$ is $\left(\sqrt{3/2}, \sqrt{3/2}, 1\right)$ in rectangular coordinates.