The Dot Product

So far we have added two vectors and multiplied a vector by a scalar. The question arises: Is it possible to multiply two vectors so that their product is a useful quantity? One such product is the dot product, whose definition follows.

1 Definition If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then the **dot product** of **a** and **b** is the number $\mathbf{a} \cdot \mathbf{b}$ given by

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Thus, to find the dot product of **a** and **b**, we multiply corresponding components and add.

The Dot Product

The result is not a vector. It is a real number, that is, a scalar. For this reason, the dot product is sometimes called the **scalar product** (or **inner product**).

Although Definition 1 is given for three-dimensional vectors, the dot product of two-dimensional vectors is defined in a similar fashion:

$$\langle a_1, a_2 \rangle \cdot \langle b_1, b_2 \rangle = a_1 b_1 + a_2 b_2$$

Example 1

$$\langle 2, 4 \rangle \cdot \langle 3, -1 \rangle = 2(3) + 4(-1)$$
$$= 2$$

$$\langle -1, 7, 4 \rangle \cdot \langle 6, 2, -\tfrac{1}{2} \rangle = (-1)(6) + 7(2) + 4(-\tfrac{1}{2})$$
$$= 6$$

$$(\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) \cdot (2\mathbf{j} - \mathbf{k}) = 1(0) + 2(2) + (-3)(-1)$$
$$= 7$$

The Dot Product

The dot product obeys many of the laws that hold for ordinary products of real numbers. These are stated in the following theorem.

2 Properties of the Dot Product If **a**, **b**, and **c** are vectors in V_3 and c is a scalar, then

1. $\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$
2. $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$
3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$
4. $(c\mathbf{a}) \cdot \mathbf{b} = c(\mathbf{a} \cdot \mathbf{b}) = \mathbf{a} \cdot (c\mathbf{b})$
5. $\mathbf{0} \cdot \mathbf{a} = 0$

The Dot Product

These properties are easily proved using Definition 1. For instance, here are the proofs of Properties 1 and 3:

1. $\mathbf{a} \cdot \mathbf{a} = a_1^2 + a_2^2 + a_3^2 = |\mathbf{a}|^2$

3. $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \langle a_1, a_2, a_3 \rangle \cdot \langle b_1 + c_1, b_2 + c_2, b_3 + c_3 \rangle$

$= a_1(b_1 + c_1) + a_2(b_2 + c_2) + a_3(b_3 + c_3)$

$= a_1b_1 + a_1c_1 + a_2b_2 + a_2c_2 + a_3b_3 + a_3c_3$

$= (a_1b_1 + a_2b_2 + a_3b_3) + (a_1c_1 + a_2c_2 + a_3c_3)$

$= \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$

The Dot Product

The dot product $\mathbf{a} \cdot \mathbf{b}$ can be given a geometric interpretation in terms of the **angle** θ **between a and b**, which is defined to be the angle between the representations of **a** and **b** that start at the origin, where $0 \le \theta \le \pi$.

In other words, θ is the angle between the line segments $\overrightarrow{OA}$ and $\overrightarrow{OB}$ in Figure 1. Note that if **a** and **b** are parallel vectors, then $\theta = 0$ or $\theta = \pi$.

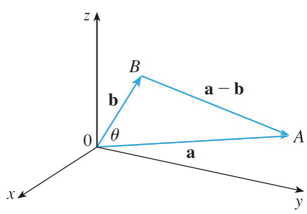


Figure 1

The Dot Product

The formula in the following theorem is used by physicists as the *definition* of the dot product.

> **3 Theorem** If θ is the angle between the vectors **a** and **b**, then
>
> $$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos \theta$$

Example 2

If the vectors **a** and **b** have lengths 4 and 6, and the angle between them is $\pi/3$, find **a** · **b**.

Solution:

Using Theorem 3, we have

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/3)$$

$$= 4 \cdot 6 \cdot \frac{1}{2}$$

$$= 12$$

The Dot Product

The formula in Theorem 3 also enables us to find the angle between two vectors.

> **6 Corollary** If θ is the angle between the nonzero vectors **a** and **b**, then
>
> $$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|}$$

Example 3

Find the angle between the vectors **a** = ⟨2, 2, –1⟩ and **b** = ⟨5, –3, 2⟩.

Solution:

Since

$$|\mathbf{a}| = \sqrt{2^2 + 2^2 + (-1)^2} = 3 \quad \text{and} \quad |\mathbf{b}| = \sqrt{5^2 + (-3)^2 + 2^2} = \sqrt{38}$$

and since

$$\mathbf{a} \cdot \mathbf{b} = 2(5) + 2(-3) + (-1)(2) = 2$$

Example 3 – *Solution*

cont'd

We have, from Corollary 6,

$$\cos \theta = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{2}{3\sqrt{38}}$$

So the angle between **a** and **b** is

$$\theta = \cos^{-1}\left(\frac{2}{3\sqrt{38}}\right)$$

$$\approx 1.46 \quad (\text{or } 84°)$$

The Dot Product

Two nonzero vectors **a** and **b** are called **perpendicular** or **orthogonal** if the angle between them is $\theta = \pi/2$. Then Theorem 3 gives

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}| \cos(\pi/2) = 0$$

and conversely if **a** · **b** = 0, then cos $\theta = 0$, so $\theta = \pi/2$. The zero vector **0** is considered to be perpendicular to all vectors.

Therefore we have the following method for determining whether two vectors are orthogonal.

> **7** Two vectors **a** and **b** are orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$.

Example 4

Show that 2**i** + 2**j** – **k** is perpendicular to 5**i** – 4**j** + 2**k**.

Solution:
Since

$$(2\mathbf{i} + 2\mathbf{j} - \mathbf{k}) \cdot (5\mathbf{i} - 4\mathbf{j} + 2\mathbf{k}) = 2(5) + 2(-4) + (-1)(2) = 0$$

these vectors are perpendicular by [7].

The Dot Product

Because cos $\theta > 0$ if $0 \leq \theta < \pi/2$ and cos $\theta < 0$ if $\pi/2 < \theta \leq \pi$, we see that **a** · **b** is positive for $\theta < \pi/2$ and negative for $\theta > \pi/2$. We can think of **a** · **b** as measuring the extent to which **a** and **b** point in the same direction.

The dot product **a** · **b** is positive if **a** and **b** point in the same general direction, 0 if they are perpendicular, and negative if they point in generally opposite directions (see Figure 2).

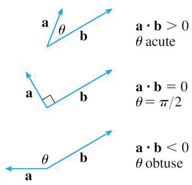


Figure 2

The Dot Product

In the extreme case where **a** and **b** point in exactly the same direction, we have $\theta = 0$, so cos $\theta = 1$ and

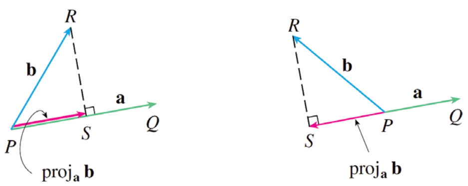
$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|$$

If **a** and **b** point in exactly opposite directions, then $\theta = \pi$ and so cos $\theta = -1$ and $\mathbf{a} \cdot \mathbf{b} = -|\mathbf{a}|\,|\mathbf{b}|$.

Projections

Projections

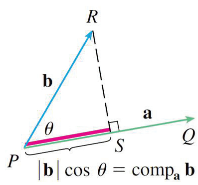
Figure 4 shows representations $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ of two vectors **a** and **b** with the same initial point P. If S is the foot of the perpendicular from R to the line containing $\overrightarrow{PQ}$, then the vector with representation $\overrightarrow{PS}$ is called the **vector projection** of **b** onto **a** and is denoted by $\text{proj}_{\mathbf{a}}\, \mathbf{b}$.
(You can think of it as a shadow of **b**).



Vector projections
Figure 4

Projections

The **scalar projection** of **b** onto **a** (also called the **component of b along a**) is defined to be the signed magnitude of the vector projection, which is the number $|\mathbf{b}| \cos \theta$, where θ is the angle between **a** and **b**.
(See Figure 5.)



Scalar projection
Figure 5

Projections

This is denoted by $\text{comp}_\mathbf{a}\,\mathbf{b}$. Observe that it is negative if $\pi/2 < \theta \leq \pi$. The equation

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}|\,|\mathbf{b}|\cos\theta = |\mathbf{a}|(|\mathbf{b}|\cos\theta)$$

shows that the dot product of **a** and **b** can be interpreted as the length of **a** times the scalar projection of **b** onto **a**. Since

$$|\mathbf{b}|\cos\theta = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{\mathbf{a}}{|\mathbf{a}|}\cdot\mathbf{b}$$

the component of **b** along **a** can be computed by taking the dot product of **b** with the unit vector in the direction of **a**.

Projections

We summarize these ideas as follows.

Scalar projection of **b** onto **a**: $\text{comp}_\mathbf{a}\,\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}$

Vector projection of **b** onto **a**: $\text{proj}_\mathbf{a}\,\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}$

Notice that the vector projection is the scalar projection times the unit vector in the direction of **a**.

Example 6

Find the scalar projection and vector projection of **b** = ⟨1, 1, 2⟩ onto **a** = ⟨–2, 3, 1⟩.

Solution:

Since $|\mathbf{a}| = \sqrt{(-2)^2 + 3^2 + 1^2} = \sqrt{14}$, the scalar projection of **b** onto **a** is

$$\text{comp}_\mathbf{a}\,\mathbf{b} = \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{(-2)(1) + 3(1) + 1(2)}{\sqrt{14}} = \frac{3}{\sqrt{14}}$$

Example 6 – *Solution*

cont'd

The vector projection is this scalar projection times the unit vector in the direction of **a**:

$$\text{proj}_\mathbf{a}\,\mathbf{b} = \frac{3}{\sqrt{14}}\,\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{3}{14}\mathbf{a} = \left\langle -\frac{3}{7}, \frac{9}{14}, \frac{3}{14}\right\rangle$$