

Return to the Heat Equation

We return now to the boundary-value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = f(x), \quad 0 < x < l; \quad u(0, t) = u(l, t) = 0 \quad (1)$$

We showed in Section 5.3 that the function

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2}$$

is a solution of the boundary-value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \quad u(0, t) = u(l, t) = 0 \quad (2)$$

for any choice of constants c_1, c_2, \dots . This led us to ask whether we can find constants c_1, c_2, \dots such that

$$u(x, 0) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right) = f(x), \quad 0 < x < l \quad (3)$$

As we showed in Section 5.5, the answer to this question is yes; if we choose

$$c_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

then the Fourier series

$$\sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right)$$

converges to $f(x)$ if f is continuous at the point x . Thus,

$$u(x, t) = \frac{2}{l} \sum_{n=1}^{\infty} \left[\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t / l^2} \quad (4)$$

is the desired solution of (1).

EXAMPLE: A thin aluminum bar ($\alpha^2 = 0.86 \text{ cm}^2/\text{s}$) 10 cm long is heated to a uniform temperature of 100°C . At time $t = 0$, the ends of the bar are plunged into an ice bath at 0°C , and thereafter they are maintained at this temperature. No heat is allowed to escape through the lateral surface of the bar. Find an expression for the temperature at any point in the bar at any later time t .

Solution: Let $u(x, t)$ denote the temperature in the bar at the point x at time t . This function satisfies the boundary-value problem

$$\frac{\partial u}{\partial t} = 0.86^2 \frac{\partial^2 u}{\partial x^2}; \quad u(x, 0) = 100, \quad 0 < x < 10; \quad u(0, t) = u(10, t) = 0 \quad (5)$$

The solution of (5) is

$$u(x, t) = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{10}\right) e^{-0.86n^2 \pi^2 t / 100}$$

where

$$\begin{aligned}c_n &= \frac{1}{5} \int_0^{10} 100 \sin\left(\frac{n\pi x}{10}\right) dx = 20 \int_0^{10} \sin\left(\frac{n\pi x}{10}\right) dx \\&= 20 \left(-\frac{10}{n\pi} \cos\left(\frac{n\pi x}{10}\right) \right) \Big|_0^{10} \\&= \frac{200}{n\pi} \left(-\cos\left(\frac{n\pi x}{10}\right) \right) \Big|_0^{10} \\&= \frac{200}{n\pi} \left(-\cos\left(\frac{n\pi(10)}{10}\right) + \cos\left(\frac{n\pi(0)}{10}\right) \right) \\&= \frac{200}{n\pi} (1 - \cos n\pi)\end{aligned}$$

Notice that $c_n = 0$ if n is even, and $c_n = 400/n\pi$ if n is odd. Hence,

$$u(x, t) = \frac{400}{\pi} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{(2n-1)\pi x}{10}\right)}{2n-1} e^{-0.86(2n-1)^2\pi^2 t/100}$$