# Even and Odd Functions

There are certain special cases when the Fourier series of a function $f$ reduces to a pure cosine or a pure sine series. These special cases occur when $f$ is even or odd.

DEFINITION: A function $f$ that satisfies

$$\boxed{f(-x) = f(x)}$$

for every number $x$ in its domain is called an **even function**. A function $f$ that satisfies

$$\boxed{f(-x) = -f(x)}$$

for every number $x$ in its domain is called an **odd function**.

REMARK: Any function is either even, or odd, or neither.

PROPERTY: Graphs of even functions are symmetric with respect to the $y$-axis. Graphs of odd functions are symmetric with respect to the origin.

EXAMPLES:

1. Functions $f(x) = x^2,\ x^4,\ x^8,\ x^4 - x^2, x^2 + 1, |x|, \cos x$, etc. are even. In fact,
   - if $f(x) = x^2$, then $f(-x) = (-x)^2 = x^2 = f(x)$
   - if $f(x) = x^4$, then $f(-x) = (-x)^4 = x^4 = f(x)$
   - if $f(x) = x^8$, then $f(-x) = (-x)^8 = x^8 = f(x)$
   - if $f(x) = x^4 - x^2$, then $f(-x) = (-x)^4 - (-x)^2 = x^4 - x^2 = f(x)$
   - if $f(x) = x^2 + 1$, then $f(-x) = (-x)^2 + 1 = x^2 + 1 = f(x)$
   - if $f(x) = |x|$, then $f(-x) = |-x| = |x| = f(x)$
   - if $f(x) = \cos x$, then $f(-x) = \cos(-x) = \cos x = f(x)$

One can see that graphs of all these functions are symmetric with respect to the $y$-axis.

2. Functions $f(x) = x,\ x^3,\ x^5,\ x^3 - x^7, \sin x$, etc. are odd. In fact,
   - if $f(x) = x$, then $f(-x) = -x = -f(x)$
   - if $f(x) = x^3$, then $f(-x) = (-x)^3 = -x^3 = -f(x)$
   - if $f(x) = x^5$, then $f(-x) = (-x)^5 = -x^5 = -f(x)$
   - if $f(x) = x^3 - x^7$, then $f(-x) = (-x)^3 - (-x)^7 = -x^3 + x^7 = -(x^3 - x^7) = -f(x)$
   - if $f(x) = \sin x$, then $f(-x) = \sin(-x) = -\sin x = -f(x)$

One can see that graphs of all these functions are symmetric with respect to the origin.

3. Functions $f(x) = x + 1,\ x^3 + x^2,\ x^5 - 2,\ |x - 2|$ etc. are neither even nor odd. In fact,
   - if $f(x) = x + 1$, then $f(-1) = -1 + 1 = 0,\ f(1) = 1 + 1 = 2$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = x^3 + x^2$, then $f(-1) = (-1)^3 + (-1)^2 = -1 + 1 = 0,\ f(1) = 1^3 + 1^2 = 2$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = x^5 - 2$, then $f(-1) = (-1)^5 - 2 = -1 - 2 = -3,\ f(1) = 1^5 - 2 = 1 - 2 = -1$, therefore $f(-1) \neq \pm f(1)$
   - if $f(x) = |x - 2|$, then $f(-1) = |-1 - 2| = |-3| = 3,\ f(1) = |1 - 2| = |-1| = 1$, therefore $f(-1) \neq \pm f(1)$

Even and odd functions satisfy the following elementary properties.

1. The product of two even functions is even.

2. The product of two odd functions is even.

3. The product of an odd function with an even function is odd.

4. The integral of an odd function $f$ over a symmetric interval $[-l, l]$ is zero; that is,

$$\int_{-l}^{l} f(x)dx = 0 \quad \text{if } f \text{ is odd}$$

5. The integral of an even function $f$ over the interval $[-l, l]$ is twice the integral of $f$ over the interval $[0, l]$; that is,

$$\int_{-l}^{l} f(x)dx = 2\int_{0}^{l} f(x)dx \quad \text{if } f \text{ is even}$$

LEMMA 1:

*(a) The Fourier series for an even function is a pure cosine series; that is, it contains no terms of the form* $\sin n\pi x/l$.

*(b) The Fourier series for an odd function is a pure sine series; that is, it contains no terms of the form* $\cos n\pi x/l$.

THEOREM 3: *Let $f$ and $f'$ be piecewise continuous on the interval $0 \le x \le l$. Compute the numbers*

$$a_n = \frac{2}{l}\int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx, \qquad n = 0, 1, 2, \ldots \tag{1}$$

*and*

$$b_n = \frac{2}{l}\int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx, \qquad n = 1, 2, \ldots \tag{2}$$

*and form the infinite series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

*and*

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

*Both series converge to $f(x)$ if $f$ is continuous at $x$, $0 < x < l$, and to*

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

*if $f$ is discontinuous at $x$, $0 < x < l$. Here the quantity $f(x+0)$ denotes the limit from the right of $f$ at the point $x$. Similarly, $f(x-0)$ denotes the limit of $f$ from the left. At $x = 0$ and $x = l$ the first series converges to $f(x)$ and the second series converges to $0$.*

EXAMPLE: Let $f(x) = 1$.

(a) Expand $f$ in a pure sine series on the interval $0 < x < \pi$.

Solution: By Theorem 3,

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\pi}\right) = \sum_{n=1}^{\infty} b_n \sin nx$$

where

$$\begin{aligned} b_n = \frac{2}{l}\int\limits_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int\limits_0^{\pi} 1\sin\left(\frac{n\pi x}{\pi}\right)dx &= \frac{2}{\pi}\int\limits_0^{\pi} \sin nx dx \\ &= \frac{2}{\pi}\left(-\frac{\cos nx}{n}\right)\Big|_0^{\pi} \\ &= \frac{2}{n\pi}\Big(-\cos n(\pi) + \cos n(0)\Big) \\ &= \frac{2}{n\pi}(1 - \cos n\pi) \\ &= \begin{cases} \dfrac{2}{n\pi}(1-1) = 0, & n \text{ even} \\ \dfrac{2}{n\pi}(1-(-1)) = \dfrac{4}{n\pi}, & n \text{ odd} \end{cases} \end{aligned}$$

Hence

$$1 = \frac{4}{\pi}\left[\frac{\sin x}{1} + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \dots\right], \qquad 0 < x < \pi$$

(b) Expand $f$ in a pure cosine series on the interval $0 \le x \le \pi$.

Solution: Putting

$$a_0 = 2, \quad a_1 = a_2 = \dots = 0$$

we get

$$1 = \frac{2}{2} + \sum_{n=1}^{\infty} 0\cos\left(\frac{n\pi x}{l}\right)$$

Note that Theorem 3 gives the same result, since

$$a_0 = \frac{2}{l}\int\limits_0^l f(x)dx = \frac{2}{\pi}\int\limits_0^{\pi} 1dx = \frac{2}{\pi}(\pi - 0) = 2$$

and

$$\begin{aligned} a_n = \frac{2}{l}\int\limits_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{\pi}\int\limits_0^{\pi} 1\cos\left(\frac{n\pi x}{\pi}\right)dx = \frac{2}{\pi}\int\limits_0^{\pi}\cos nx dx &= \frac{2}{\pi}\left(\frac{\sin nx}{n}\right)\Big|_0^{\pi} \\ &= \frac{2}{n\pi}\Big(\sin n(\pi) - \sin n(0)\Big) \\ &= \frac{2}{n\pi}(0-0) \\ &= 0 \end{aligned}$$

EXAMPLE: Expand the function $f(x) = e^x$ in a pure cosine series on the interval $0 \le x \le 1$.

Solution: By Theorem 3,

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

$$= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{1}\right) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x$$

where

$$a_0 = \frac{2}{l}\int_0^l f(x)dx = 2\int_0^1 e^x dx = 2e^x\Big|_0^1 = 2(e^1 - e^0) = 2(e-1)$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{1}\int_0^1 e^x \cos\left(\frac{n\pi x}{1}\right)dx$$

$$= 2\int_0^1 e^x \cos n\pi x dx$$

$$= 2\,\mathrm{Re}\int_0^1 e^x e^{in\pi x} dx$$

$$= 2\,\mathrm{Re}\int_0^1 e^{(1+in\pi)x} dx$$

$$= 2\,\mathrm{Re}\left\{\frac{e^{(1+in\pi)x}}{1+in\pi}\Big|_0^1\right\}$$

$$= 2\,\mathrm{Re}\left\{\frac{e^{(1+in\pi)(1)} - e^{(1+in\pi)(0)}}{1+in\pi}\right\}$$

$$= 2\,\mathrm{Re}\left\{\frac{e^{1+in\pi} - 1}{1+in\pi}\right\}$$

One can show (see Appendix I) that

$$2\,\mathrm{Re}\left\{\frac{e^{1+in\pi} - 1}{1+in\pi}\right\} = \frac{2(e\cos n\pi - 1)}{1+n^2\pi^2}$$

Hence

$$e^x = e - 1 + 2\sum_{n=1}^{\infty} \frac{(e\cos n\pi - 1)}{1+n^2\pi^2}\cos n\pi x, \qquad 0 \le x \le 1$$

REMARK: Another way to find $\int_0^1 e^x \cos n\pi x dx$ is shown in Appendix II.

EXAMPLE: Let

$$f(x)=\begin{cases} 0 & \text{for } 0\le x\le 1 \\ 1 & \text{for } 1<x\le 2 \end{cases}$$

Expand the function $f(x)$ in a pure cosine series on the interval $0\le x\le 2$.

Solution: By Theorem 3,

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{2}\right)$$

where

$$\begin{aligned}
a_0&=\frac{2}{l}\int_0^l f(x)dx\\
&=\frac{2}{2}\int_0^2 f(x)dx\\
&=\int_0^2 f(x)dx\\
&=\int_0^1 f(x)dx+\int_1^2 f(x)dx\\
&=\int_0^1 0dx+\int_1^2 1dx\\
&=\int_1^2 1dx\\
&=x\Big|_1^2\\
&=2-1\\
&=1
\end{aligned}$$

and

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx = \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx = \int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 (0)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 (1)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_1^2 \cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right)\Bigg|_1^2$$

$$= \frac{2}{n\pi}\left(\sin\left(\frac{n\pi(2)}{2}\right) - \sin\left(\frac{n\pi(1)}{2}\right)\right)$$

$$= \frac{2}{n\pi}\left(\sin n\pi - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \frac{2}{n\pi}\left(0 - \sin\left(\frac{n\pi}{2}\right)\right)$$

$$= \begin{cases} \dfrac{2}{n\pi}(0-0) = 0 & \text{if } n \text{ is even} \\ \dfrac{2}{n\pi}(0-1) = -\dfrac{2}{n\pi} & \text{if } (n-1)/2 \text{ is even} \\ \dfrac{2}{n\pi}(0-(-1)) = \dfrac{2}{n\pi} & \text{if } (n-1)/2 \text{ is odd} \end{cases}$$

Hence

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) = \frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\cos\left(\frac{(2k-1)\pi x}{2}\right)$$

if $0 \le x \le 2$ and $x \neq 1$. At $x = 1$ this series reduces to the single number $\frac{1}{2}$, since

$$\frac{1}{2} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k}{2k-1}\cos\left(\frac{(2k-1)\pi(1)}{2}\right) = \frac{1}{2} + 0 + 0 + 0 + \ldots = \frac{1}{2}$$

Note that $\frac{1}{2}$ is the value predicted by Theorem 3, since

$$\frac{1}{2}\left[\lim_{x\to 1^+} f(x) + \lim_{x\to 1^-} f(x)\right] = \frac{1}{2}[1+0] = \frac{1}{2}$$

# Appendix I

Here we show that

$$\mathrm{Re}\left\{\frac{e^{1+in\pi}-1}{1+in\pi}\right\} = \frac{e\cos n\pi - 1}{1+n^2\pi^2} \tag{3}$$

To this end, we first note that

$$\begin{aligned}
\frac{e^{1+in\pi}-1}{1+in\pi} = \frac{e\cdot e^{in\pi}-1}{1+in\pi} = \frac{e(\cos n\pi + i\sin n\pi)-1}{1+in\pi} &= \frac{e(\cos n\pi + i\cdot 0)-1}{1+in\pi} \\
&= \frac{e\cos n\pi - 1}{1+in\pi} \\
&= \frac{(e\cos n\pi - 1)(1-in\pi)}{(1+in\pi)(1-in\pi)} \\
&= \frac{(e\cos n\pi - 1)(1-in\pi)}{1^2-(in\pi)^2} \\
&= \frac{(e\cos n\pi - 1)(1-in\pi)}{1+n^2\pi^2}
\end{aligned}$$

so

$$\mathrm{Re}\left\{\frac{e^{1+in\pi}-1}{1+in\pi}\right\} = \mathrm{Re}\left\{\frac{(e\cos n\pi - 1)(1-in\pi)}{1+n^2\pi^2}\right\} \tag{4}$$

But

$$(e\cos n\pi - 1)(1-in\pi) = (1)(e\cos n\pi - 1) - (in\pi)(e\cos n\pi - 1)$$

therefore

$$\mathrm{Re}\left\{(e\cos n\pi - 1)(1-in\pi)\right\} = e\cos n\pi - 1$$

This and (4) give (3).

# Appendix II

Here we show that

$$\int_0^1 e^x \cos n\pi x dx = \frac{e\cos n\pi - 1}{1+n^2\pi^2}$$

By the integration by parts formula

$$\int u dv = uv - \int v du$$

we get

$$\int e^x \cos n\pi x dx = \left[ \begin{array}{c|c} e^x = u & \cos n\pi x dx = dv \\ e^x dx = du & \frac{1}{n\pi}\sin n\pi x = v \end{array} \right] = e^x \cdot \frac{1}{n\pi}\sin n\pi x - \int \frac{1}{n\pi}\sin n\pi x \cdot e^x dx$$

$$= \frac{1}{n\pi}e^x \sin n\pi x - \frac{1}{n\pi}\int e^x \sin n\pi x dx = \left[ \begin{array}{c|c} e^x = u & \sin n\pi x dx = dv \\ e^x dx = du & -\frac{1}{n\pi}\cos n\pi x = v \end{array} \right]$$

$$= \frac{1}{n\pi}e^x \sin n\pi x - \frac{1}{n\pi}\left(-e^x \cdot \frac{1}{n\pi}\cos n\pi x + \int \frac{1}{n\pi}\cos n\pi x \cdot e^x dx\right)$$

$$= \frac{1}{n\pi}e^x \sin n\pi x - \frac{1}{n\pi}\left(-e^x \cdot \frac{1}{n\pi}\cos n\pi x + \frac{1}{n\pi}\int e^x \cos n\pi x dx\right)$$

$$= \frac{1}{n\pi}e^x \sin n\pi x + \frac{1}{n^2\pi^2}e^x \cos n\pi x - \frac{1}{n^2\pi^2}\int e^x \cos n\pi x dx$$

so

$$\int e^x \cos n\pi x dx = \frac{1}{n\pi}e^x \sin n\pi x + \frac{1}{n^2\pi^2}e^x \cos n\pi x - \frac{1}{n^2\pi^2}\int e^x \cos n\pi x dx$$

hence

$$n^2\pi^2 \int e^x \cos n\pi x dx = n\pi e^x \sin n\pi x + e^x \cos n\pi x - \int e^x \cos n\pi x dx$$

$$\int e^x \cos n\pi x dx + n^2\pi^2 \int e^x \cos n\pi x dx = n\pi e^x \sin n\pi x + e^x \cos n\pi x + C_1$$

$$(1+n^2\pi^2)\int e^x \cos n\pi x dx = n\pi e^x \sin n\pi x + e^x \cos n\pi x + C_1$$

$$\int e^x \cos n\pi x dx = \frac{1}{1+n^2\pi^2}\left(n\pi e^x \sin n\pi x + e^x \cos n\pi x\right) + C$$

Therefore

$$\int_0^1 e^x \cos n\pi x dx = \frac{1}{1+n^2\pi^2}\left(n\pi e^x \sin n\pi x + e^x \cos n\pi x\right)\Big|_0^1$$

$$= \frac{1}{1+n^2\pi^2}\left[\left(n\pi e^1 \sin n\pi(1) + e^1 \cos n\pi(1)\right) - \left(n\pi e^0 \sin n\pi(0) + e^0 \cos n\pi(0)\right)\right]$$

$$= \frac{1}{1+n^2\pi^2}\left[(n\pi e \sin n\pi + e\cos n\pi) - (n\pi \sin 0 + \cos 0)\right]$$

$$= \frac{1}{1+n^2\pi^2}\left[(0 + e\cos n\pi) - (0+1)\right]$$

$$= \frac{e\cos n\pi - 1}{1+n^2\pi^2}$$