Fourier Series

THEOREM: *Let f and f' be piecewise continuous on the interval $-l \leq x \leq l$. (This means that f and f' have only a finite number of discontinuities on this interval, and both f and f' have right- and left-hand limits at each point of discontinuity.) Compute the numbers a_n and b_n from*

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}dx, \quad n = 0, 1, 2, \ldots \tag{1}$$

and

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}dx, \quad n = 1, 2, \ldots \tag{2}$$

and form the infinite series

$$\frac{a_0}{2} + a_1\cos\frac{\pi x}{l} + b_1\sin\frac{\pi x}{l} + \ldots = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left[a_n\cos\frac{n\pi x}{l} + b_n\sin\frac{n\pi x}{l}\right] \tag{3}$$

This series, which is called the ***Fourier series*** *for f on the interval $-l \leq x \leq l$, converges to $f(x)$ if f is continuous at x, and to*

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

if f is discontinuous at x. At $x = \pm l$, the Fourier series (3) converges to

$$\frac{1}{2}[f(l) + f(-l)]$$

where $f(\pm l)$ is the limit of $f(x)$ as x approaches $\pm l$.

REMARK 1: The quantity

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

is the average of the right- and left-hand limits of f at the point x. If we define $f(x)$ to be the average of the right- and left-hand limits of f at any point of discontinuity x, then the Fourier series (3) converges to $f(x)$ for all points x in the interval $-l < x < l$.

REMARK 2: A function f can be expanded in one, and only one, Fourier series on the interval $-l < x < l$.

EXAMPLE: Let

$$f(x) = \begin{cases} 0 & \text{for } \; -1 \leq x < 0 \\ 1 & \text{for } \; 0 \leq x \leq 1 \end{cases}$$

Compute the Fourier series for f on the interval $-1 \leq x \leq 1$.

Solution: In this problem, $l = 1$. Hence, from (1) and (2),

$$a_0 = \int_{-1}^{1} f(x)dx = \int_{-1}^{0} f(x)dx + \int_{0}^{1} f(x)dx$$

$$= \int_{-1}^{0} 0dx + \int_{0}^{1} 1dx = \int_{0}^{1} dx = 1$$

$$a_n = \int_{-1}^{1} f(x)\cos n\pi x dx = \int_{-1}^{0} f(x)\cos n\pi x dx + \int_{0}^{1} f(x)\cos n\pi x dx$$

$$= \int_{-1}^{0} (0)\cos n\pi x dx + \int_{0}^{1} (1)\cos n\pi x dx = \int_{0}^{1} \cos n\pi x dx = 0, \quad n \geq 1$$

and

$$b_n = \int_{-1}^{1} f(x)\sin n\pi x dx = \int_{-1}^{0} f(x)\sin n\pi x dx + \int_{0}^{1} f(x)\sin n\pi x dx$$

$$= \int_{-1}^{0} (0)\sin n\pi x dx + \int_{0}^{1} (1)\sin n\pi x dx$$

$$= \int_{0}^{1} \sin n\pi x dx = \frac{1}{n\pi}(1 - \cos n\pi) = \frac{1-(-1)^n}{n\pi}, \quad n \geq 1$$

Notice that $b_n = 0$ for n even, and $b_n = 2/n\pi$ for n odd. Hence, the Fourier series for f on the interval $-1 \leq x \leq 1$ is

$$\begin{aligned}
f(x) = \frac{a_0}{2} &+ a_1 \cos\frac{\pi x}{l} + b_1 \sin\frac{\pi x}{l} + a_2 \cos\frac{2\pi x}{l} + b_2 \sin\frac{2\pi x}{l} \\
&+ a_3 \cos\frac{3\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + a_4 \cos\frac{4\pi x}{l} + b_4 \sin\frac{4\pi x}{l} \\
&+ a_5 \cos\frac{5\pi x}{l} + b_5 \sin\frac{5\pi x}{l} + a_6 \cos\frac{6\pi x}{l} + b_6 \sin\frac{6\pi x}{l} + \ldots \\
= \frac{a_0}{2} &+ 0\cos\frac{\pi x}{1} + b_1 \sin\frac{\pi x}{1} + 0\cos\frac{2\pi x}{1} + 0\sin\frac{2\pi x}{1} \\
&+ 0\cos\frac{3\pi x}{1} + b_3 \sin\frac{3\pi x}{1} + 0\cos\frac{4\pi x}{1} + 0\sin\frac{4\pi x}{1} \\
&+ 0\cos\frac{5\pi x}{1} + b_5 \sin\frac{5\pi x}{1} + 0\cos\frac{6\pi x}{1} + 0\sin\frac{6\pi x}{1} + \ldots \\
= \frac{a_0}{2} &+ b_1 \sin\pi x + b_3 \sin 3\pi x + b_5 \sin 5\pi x + \ldots \\
= \frac{1}{2} &+ \frac{2\sin\pi x}{\pi} + \frac{2\sin 3\pi x}{3\pi} + \frac{2\sin 5\pi x}{5\pi} + \ldots
\end{aligned}$$

By the Theorem above, this series converges to 0 if $-1 < x < 0$, and to 1 if $0 < x < 1$. At $x = -1, 0$, and 1, this series reduces to the single number $\frac{1}{2}$, which is the value predicted for it by the Theorem.

EXAMPLE: Let

$$f(x)=\begin{cases} 1 & \text{for} \quad -2\le x<0 \\ x & \text{for} \quad 0\le x\le 2 \end{cases}$$

Compute the Fourier series for f on the interval $-2\le x\le 2$.

Solution: In this problem, $l=2$. Hence, from (1) and (2),

$$a_0=\frac{1}{2}\int\limits_{-2}^{2} f(x)dx=\frac{1}{2}\int\limits_{-2}^{0} dx+\frac{1}{2}\int\limits_{0}^{2} xdx=2$$

and (see the Appendix)

$$a_n=\frac{1}{2}\int\limits_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx=\frac{1}{2}\int\limits_{-2}^{0}\cos\frac{n\pi x}{2}dx+\frac{1}{2}\int\limits_{0}^{2} x\cos\frac{n\pi x}{2}dx=\frac{2}{(n\pi)^2}(\cos n\pi-1), \qquad n\ge 1$$

$$b_n=\frac{1}{2}\int\limits_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx=\frac{1}{2}\int\limits_{-2}^{0}\sin\frac{n\pi x}{2}dx+\frac{1}{2}\int\limits_{0}^{2} x\sin\frac{n\pi x}{2}dx=-\frac{1}{n\pi}(1+\cos n\pi), \qquad n\ge 1$$

Notice that $a_n=0$ if $n>0$ is even; $a_n=-4/n^2\pi^2$ if n is odd; $b_n=0$ if n is odd; and $b_n=-2/n\pi$ if n is even. Hence, the Fourier series for f on the interval $-2\le x\le 2$ is

$$\begin{aligned} f(x) &= \frac{a_0}{2}+a_1\cos\frac{\pi x}{l}+b_1\sin\frac{\pi x}{l}+a_2\cos\frac{2\pi x}{l}+b_2\sin\frac{2\pi x}{l} \\ &\qquad +a_3\cos\frac{3\pi x}{l}+b_3\sin\frac{3\pi x}{l}+a_4\cos\frac{4\pi x}{l}+b_4\sin\frac{4\pi x}{l}\ldots \\ &= \frac{a_0}{2}+a_1\cos\frac{\pi x}{2}+0\sin\frac{\pi x}{2}+0\cos\frac{2\pi x}{2}+b_2\sin\frac{2\pi x}{2} \\ &\qquad +a_3\cos\frac{3\pi x}{2}+0\sin\frac{3\pi x}{2}+0\cos\frac{4\pi x}{2}+b_4\sin\frac{4\pi x}{2}\ldots \\ &= \frac{a_0}{2}+a_1\cos\frac{\pi x}{2}+b_2\sin\frac{2\pi x}{2}+a_3\cos\frac{3\pi x}{2}+b_4\sin\frac{4\pi x}{2}+\ldots \\ &= 1-\frac{4}{\pi^2}\cos\frac{\pi x}{2}-\frac{1}{\pi}\sin\pi x-\frac{4}{9\pi^2}\cos\frac{3\pi x}{2}-\frac{1}{2\pi}\sin 2\pi x-\ldots \\ &= 1-\frac{4}{\pi^2}\sum_{n=0}^{\infty}\frac{\cos\frac{(2n+1)\pi x}{2}}{(2n+1)^2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\pi x}{n} \qquad (4) \end{aligned}$$

By the Theorem above, this series converges to 1 if $-2<x<0$; to x if $0<x<2$; to $\frac{1}{2}$ if $x=0$; and to $\frac{3}{2}$ if $x=\pm 2$. Now, at $x=0$, the Fourier series (4) is

$$1-\frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right]$$

Thus, we deduce the remarkable identity

$$\frac{1}{2}=1-\frac{4}{\pi^2}\left[\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots\right]$$

or

$$\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\ldots=\frac{\pi^2}{8}$$

EXAMPLE: Find the Fourier series for the function $f(x) = \cos^2 x$ on the interval $-\pi \le x \le \pi$.

Solution: By Remark 2, the function $f(x) = \cos^2 x$ has a unique Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\pi} + b_n \sin \frac{n\pi x}{\pi} \right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

on the interval $-\pi \le x \le \pi$. But we already know that

$$\cos^2 x = \frac{1}{2} + \frac{\cos 2x}{2}$$

Hence, the Fourier series for $\cos^2 x$ on the interval $-\pi \le x \le \pi$ must be

$$\frac{1}{2} + \frac{1}{2} \cos 2x$$

that is,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = a_4 = \ldots = 0$$

and

$$b_1 = b_2 = b_3 = \ldots = 0$$

Appendix

We have

$$\int_{-2}^{0} \cos\frac{n\pi x}{2}dx = \frac{2}{n\pi}\sin\frac{n\pi x}{2}\bigg|_{-2}^{0} = \frac{2}{n\pi}\left(\sin\frac{n\pi(0)}{2} - \sin\frac{n\pi(-2)}{2}\right)$$

$$= \frac{2}{n\pi}\left(\sin 0 - \sin(-n\pi)\right)$$

$$= \frac{2}{n\pi}\left(0 - 0\right)$$

$$= 0$$

and

$$\int_{0}^{2} x\cos\frac{n\pi x}{2}dx = \left[\begin{array}{c|c} x = u & \cos\frac{n\pi x}{2}dx = dv \\ dx = du & \frac{2}{n\pi}\sin\frac{n\pi x}{2} = v \end{array}\right] = x\frac{2}{n\pi}\sin\frac{n\pi x}{2}\bigg|_{0}^{2} - \int_{0}^{2}\frac{2}{n\pi}\sin\frac{n\pi x}{2}dx$$

$$= \left(\frac{4}{n\pi}\sin n\pi - 0\right) - \frac{2}{n\pi}\int_{0}^{2}\sin\frac{n\pi x}{2}dx$$

$$= -\frac{2}{n\pi}\int_{0}^{2}\sin\frac{n\pi x}{2}dx$$

$$= \left(-\frac{2}{n\pi}\right)\left(-\frac{2}{n\pi}\cos\frac{n\pi x}{2}\right)\bigg|_{0}^{2}$$

$$= \frac{4}{(n\pi)^2}\left(\cos n\pi - \cos 0\right)$$

$$= \frac{4}{(n\pi)^2}\left(\cos n\pi - 1\right)$$

therefore

$$\frac{1}{2}\int_{-2}^{2} f(x)\cos\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-2}^{0}\cos\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} x\cos\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\cdot 0 + \frac{1}{2}\cdot\frac{4}{(n\pi)^2}\left(\cos n\pi - 1\right) = \frac{2}{(n\pi)^2}(\cos n\pi - 1)$$

We have

$$\int_{-2}^{0} \sin\frac{n\pi x}{2}dx = -\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{-2}^{0} = -\frac{2}{n\pi}\left(\cos\frac{n\pi(0)}{2} - \cos\frac{n\pi(-2)}{2}\right)$$

$$= -\frac{2}{n\pi}\left(\cos 0 - \cos(-n\pi)\right)$$

$$= -\frac{2}{n\pi}\left(1 - \cos n\pi\right)$$

and

$$\int_{0}^{2} x\sin\frac{n\pi x}{2}dx = \left[\begin{array}{c|c} x = u & \sin\frac{n\pi x}{2}dx = dv \\ dx = du & -\frac{2}{n\pi}\cos\frac{n\pi x}{2} = v \end{array}\right] = -x\frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_{0}^{2} - \int_{0}^{2}\left(-\frac{2}{n\pi}\right)\cos\frac{n\pi x}{2}dx$$

$$= \left(-\frac{4}{n\pi}\cos n\pi + 0\right) + \frac{2}{n\pi}\int_{0}^{2}\cos\frac{n\pi x}{2}dx$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\int_{0}^{2}\cos\frac{n\pi x}{2}dx$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{2}{n\pi}\cdot\frac{2}{n\pi}\sin\frac{n\pi x}{2}\Big|_{0}^{2}$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{4}{(n\pi)^2}\left(\sin n\pi - \sin 0\right)$$

$$= -\frac{4}{n\pi}\cos n\pi + \frac{4}{(n\pi)^2}\left(0 - 0\right)$$

$$= -\frac{4}{n\pi}\cos n\pi$$

therefore

$$\frac{1}{2}\int_{-2}^{2} f(x)\sin\frac{n\pi x}{2}dx = \frac{1}{2}\int_{-2}^{0}\sin\frac{n\pi x}{2}dx + \frac{1}{2}\int_{0}^{2} x\sin\frac{n\pi x}{2}dx$$

$$= \frac{1}{2}\cdot\left(-\frac{2}{n\pi}\left(1 - \cos n\pi\right)\right) + \frac{1}{2}\cdot\left(-\frac{4}{n\pi}\cos n\pi\right)$$

$$= -\frac{1}{n\pi}\left(1 - \cos n\pi\right) - \frac{2}{n\pi}\cos n\pi$$

$$= -\frac{1}{n\pi} + \frac{1}{n\pi}\cos n\pi - \frac{2}{n\pi}\cos n\pi$$

$$= -\frac{1}{n\pi} - \frac{1}{n\pi}\cos n\pi$$

$$= -\frac{1}{n\pi}(1 + \cos n\pi)$$