

Fourier Series

THEOREM: Let f and f' be piecewise continuous on the interval $-l \leq x \leq l$. (This means that f and f' have only a finite number of discontinuities on this interval, and both f and f' have right- and left-hand limits at each point of discontinuity.) Compute the numbers

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos \frac{n\pi x}{l} dx, \quad n = 0, 1, 2, \dots \quad (1)$$

and

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin \frac{n\pi x}{l} dx, \quad n = 1, 2, \dots \quad (2)$$

and form the infinite series

$$\frac{a_0}{2} + a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + \dots = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{l} + b_n \sin \frac{n\pi x}{l} \right] \quad (3)$$

This series is called the **Fourier series** for f on the interval $-l \leq x \leq l$. It converges to $f(x)$ if f is continuous at x , $-l < x < l$, and to

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

if f is discontinuous at x , $-l < x < l$. Here the quantity $f(x+0)$ denotes the limit from the right of f at the point x . Similarly, $f(x-0)$ denotes the limit of f from the left. At $x = \pm l$, the Fourier series (3) converges to

$$\frac{1}{2}[f(l) + f(-l)]$$

where $f(\pm l)$ is the limit of $f(x)$ as x approaches $\pm l$. That is, $\lim_{x \rightarrow l^+} f(x)$ and $\lim_{x \rightarrow l^-} f(x)$, respectively.

REMARK 1: The quantity

$$\frac{1}{2}[f(x+0) + f(x-0)]$$

is the average of the right- and left-hand limits of f at the point x . If we define $f(x)$ to be the average of the right- and left-hand limits of f at any point of discontinuity x , then the Fourier series (3) converges to $f(x)$ for all points x in the interval $-l < x < l$.

REMARK 2: A function f can be expanded in one, and only one, Fourier series on the interval $-l < x < l$.

EXAMPLE: Let

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 \leq x \leq 1 \end{cases}$$

Compute the Fourier series for f on the interval $-1 \leq x \leq 1$.

Solution: In this problem, $l = 1$. Hence, from (1) and (2),

$$\begin{aligned} a_0 &= \int_{-1}^1 f(x) dx = \int_{-1}^0 f(x) dx + \int_0^1 f(x) dx \\ &= \int_{-1}^0 0 dx + \int_0^1 1 dx = \int_0^1 1 dx = x \Big|_0^1 = 1 - 0 = 1 \end{aligned}$$

For any $n \geq 1$ we have

$$\begin{aligned} a_n &= \int_{-1}^1 f(x) \cos n\pi x dx = \int_{-1}^0 f(x) \cos n\pi x dx + \int_0^1 f(x) \cos n\pi x dx \\ &= \int_{-1}^0 (0) \cos n\pi x dx + \int_0^1 (1) \cos n\pi x dx \\ &= \int_0^1 \cos n\pi x dx \\ &= \frac{1}{n\pi} \sin n\pi x \Big|_0^1 \\ &= \frac{1}{n\pi} (\sin n\pi - \sin 0) \\ &= \frac{1}{n\pi} (0 - 0) \\ &= 0 \end{aligned}$$

and

$$\begin{aligned} b_n &= \int_{-1}^1 f(x) \sin n\pi x dx = \int_{-1}^0 f(x) \sin n\pi x dx + \int_0^1 f(x) \sin n\pi x dx \\ &= \int_{-1}^0 (0) \sin n\pi x dx + \int_0^1 (1) \sin n\pi x dx \\ &= \int_0^1 \sin n\pi x dx \\ &= -\frac{1}{n\pi} \cos n\pi x \Big|_0^1 \\ &= -\frac{1}{n\pi} (\cos n\pi - \cos 0) \\ &= \begin{cases} -\frac{1}{n\pi} (1 - 1) = 0 & \text{if } n \text{ is even} \\ -\frac{1}{n\pi} ((-1) - 1) = \frac{2}{n\pi} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

Hence, the Fourier series for f on the interval $-1 \leq x \leq 1$ is

$$\begin{aligned}
& \frac{a_0}{2} + a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} \\
& + a_3 \cos \frac{3\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + a_4 \cos \frac{4\pi x}{l} + b_4 \sin \frac{4\pi x}{l} \\
& + a_5 \cos \frac{5\pi x}{l} + b_5 \sin \frac{5\pi x}{l} + a_6 \cos \frac{6\pi x}{l} + b_6 \sin \frac{6\pi x}{l} + \dots \\
& = \frac{1}{2} + 0 \cos \frac{\pi x}{1} + b_1 \sin \frac{\pi x}{1} + 0 \cos \frac{2\pi x}{1} + 0 \sin \frac{2\pi x}{1} \\
& + 0 \cos \frac{3\pi x}{1} + b_3 \sin \frac{3\pi x}{1} + 0 \cos \frac{4\pi x}{1} + 0 \sin \frac{4\pi x}{1} \\
& + 0 \cos \frac{5\pi x}{1} + b_5 \sin \frac{5\pi x}{1} + 0 \cos \frac{6\pi x}{1} + 0 \sin \frac{6\pi x}{1} + \dots \\
& = \frac{1}{2} + b_1 \sin \pi x + b_3 \sin 3\pi x + b_5 \sin 5\pi x + \dots \\
& = \frac{1}{2} + \frac{2 \sin \pi x}{\pi} + \frac{2 \sin 3\pi x}{3\pi} + \frac{2 \sin 5\pi x}{5\pi} + \dots = \frac{1}{2} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}
\end{aligned}$$

By the Theorem above, this series converges to $f(x)$ if f is continuous at x . That is, the series converges to

$$0 \quad \text{if} \quad -1 < x < 0$$

and to

$$1 \quad \text{if} \quad 0 < x < 1$$

At $x = 0$ and ± 1 this series reduces to the single number $\frac{1}{2}$, since

$$\begin{aligned}
\frac{1}{2} + \frac{2 \sin(\pi(0))}{\pi} + \frac{2 \sin(3\pi(0))}{3\pi} + \frac{2 \sin(5\pi(0))}{5\pi} + \dots &= \frac{1}{2} + 0 + 0 + 0 + \dots = \frac{1}{2} \\
\frac{1}{2} + \frac{2 \sin(\pi(1))}{\pi} + \frac{2 \sin(3\pi(1))}{3\pi} + \frac{2 \sin(5\pi(1))}{5\pi} + \dots &= \frac{1}{2} + 0 + 0 + 0 + \dots = \frac{1}{2} \\
\frac{1}{2} + \frac{2 \sin(\pi(-1))}{\pi} + \frac{2 \sin(3\pi(-1))}{3\pi} + \frac{2 \sin(5\pi(-1))}{5\pi} + \dots &= \frac{1}{2} + 0 + 0 + 0 + \dots = \frac{1}{2}
\end{aligned}$$

Note that $\frac{1}{2}$ is the value predicted by the Theorem above, since

(a) the series converges to

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2}[1 + 0] = \frac{1}{2}$$

because $x = 0$ is the point where f is discontinuous;

(b) the series converges to

$$\frac{1}{2}[f(1) + f(-1)] = \frac{1}{2}[1 + 0] = \frac{1}{2}$$

because $x = \pm 1$ are the end points.

EXAMPLE: Let

$$f(x) = \begin{cases} 1 & \text{for } -2 \leq x < 0 \\ x & \text{for } 0 \leq x \leq 2 \end{cases}$$

Compute the Fourier series for f on the interval $-2 \leq x \leq 2$.

Solution: In this problem, $l = 2$. Hence, from (1) and (2),

$$a_0 = \frac{1}{2} \int_{-2}^2 f(x) dx = \frac{1}{2} \int_{-2}^0 1 dx + \frac{1}{2} \int_0^2 x dx = \frac{1}{2} x \Big|_{-2}^0 + \frac{1}{2} \frac{x^2}{2} \Big|_0^2 = \frac{1}{2}(0 + 2) + \frac{1}{2} \left(\frac{2^2}{2} - \frac{0^2}{2} \right) = 2$$

and (see Appendix II)

$$a_n = \frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-2}^0 \cos \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx = \frac{2}{(n\pi)^2} (\cos n\pi - 1), \quad n \geq 1$$

$$b_n = \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx = \frac{1}{2} \int_{-2}^0 \sin \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \sin \frac{n\pi x}{2} dx = -\frac{1}{n\pi} (1 + \cos n\pi), \quad n \geq 1$$

Notice that $a_n = 0$ if $n > 0$ is even; $a_n = -4/n^2\pi^2$ if n is odd; $b_n = 0$ if n is odd; and $b_n = -2/n\pi$ if n is even. Hence, the Fourier series for f on the interval $-2 \leq x \leq 2$ is

$$\begin{aligned} & \frac{a_0}{2} + a_1 \cos \frac{\pi x}{l} + b_1 \sin \frac{\pi x}{l} + a_2 \cos \frac{2\pi x}{l} + b_2 \sin \frac{2\pi x}{l} \\ & + a_3 \cos \frac{3\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + a_4 \cos \frac{4\pi x}{l} + b_4 \sin \frac{4\pi x}{l} \\ & + a_5 \cos \frac{5\pi x}{l} + b_5 \sin \frac{5\pi x}{l} + a_6 \cos \frac{6\pi x}{l} + b_6 \sin \frac{6\pi x}{l} \dots \\ = & \frac{2}{2} + a_1 \cos \frac{\pi x}{2} + 0 \sin \frac{\pi x}{2} + 0 \cos \frac{2\pi x}{2} + b_2 \sin \frac{2\pi x}{2} \\ & + a_3 \cos \frac{3\pi x}{2} + 0 \sin \frac{3\pi x}{2} + 0 \cos \frac{4\pi x}{2} + b_4 \sin \frac{4\pi x}{2} \\ & + a_5 \cos \frac{5\pi x}{2} + 0 \sin \frac{5\pi x}{2} + 0 \cos \frac{6\pi x}{2} + b_6 \sin \frac{6\pi x}{2} \dots \\ = & 1 + a_1 \cos \frac{\pi x}{2} + b_2 \sin \frac{2\pi x}{2} + a_3 \cos \frac{3\pi x}{2} + b_4 \sin \frac{4\pi x}{2} + a_5 \cos \frac{5\pi x}{2} + b_6 \sin \frac{6\pi x}{2} + \dots \\ = & 1 - \frac{4}{\pi^2} \cos \frac{\pi x}{2} - \frac{1}{\pi} \sin \pi x - \frac{4}{9\pi^2} \cos \frac{3\pi x}{2} - \frac{1}{2\pi} \sin 2\pi x - \frac{4}{25\pi^2} \cos \frac{5\pi x}{2} - \frac{1}{3\pi} \sin 3\pi x - \dots \\ = & 1 - \frac{4}{\pi^2} \cos \frac{\pi x}{2} - \frac{4}{9\pi^2} \cos \frac{3\pi x}{2} - \frac{4}{25\pi^2} \cos \frac{5\pi x}{2} - \dots - \frac{1}{\pi} \sin \pi x - \frac{1}{2\pi} \sin 2\pi x - \frac{1}{3\pi} \sin 3\pi x - \dots \\ = & 1 - \frac{4}{\pi^2} \left(\cos \frac{\pi x}{2} + \frac{1}{9} \cos \frac{3\pi x}{2} + \frac{1}{25} \cos \frac{5\pi x}{2} + \dots \right) - \frac{1}{\pi} \left(\sin \pi x + \frac{1}{2} \sin 2\pi x + \frac{1}{3} \sin 3\pi x + \dots \right) \\ = & 1 - \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{\cos \frac{(2n-1)\pi x}{2}}{(2n-1)^2} - \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin n\pi x}{n} \end{aligned} \tag{4}$$

By the Theorem above, this series converges to $f(x)$ if f is continuous at x . That is, the series converges to

$$1 \quad \text{if} \quad -2 < x < 0$$

and to

$$x \quad \text{if} \quad 0 < x < 2$$

At $x = 0$ this series reduces to the single number $\frac{1}{2}$ by the Theorem above. Indeed, since $x = 0$ is the point where f is discontinuous, we have

$$\frac{1}{2}[f(x+0) + f(x-0)] = \frac{1}{2}[0 + 1] = \frac{1}{2}$$

Finally, the series converges to

$$\frac{1}{2}[f(2) + f(-2)] = \frac{1}{2}[2 + 1] = \frac{3}{2}$$

because $x = \pm 2$ are the end points.

NOTE: At $x = 0$ the Fourier series (4) is

$$1 - \frac{4}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$$

Thus, we deduce the remarkable identity

$$\frac{1}{2} = 1 - \frac{4}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots \right]$$

which gives

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \dots = \frac{\pi^2}{8}$$

EXAMPLE: Find the Fourier series for the function $f(x) = \cos^2 x$ on the interval $-\pi \leq x \leq \pi$.

Solution: By Remark 2, the function $f(x) = \cos^2 x$ has a unique Fourier series

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \frac{n\pi x}{\pi} + b_n \sin \frac{n\pi x}{\pi} \right] = \frac{a_0}{2} + \sum_{n=1}^{\infty} [a_n \cos nx + b_n \sin nx]$$

on the interval $-\pi \leq x \leq \pi$. But we already know that

$$\begin{aligned} \cos^2 x &= \frac{1}{2} + \frac{1}{2} \cos 2x \\ &= \frac{1}{2} + (0) \cos x + (0) \sin x + \frac{1}{2} \cos 2x + (0) \sin 2x + (0) \cos 3x + (0) \sin 3x + \dots \end{aligned}$$

Hence, the Fourier series for $\cos^2 x$ on the interval $-\pi \leq x \leq \pi$ must be

$$\frac{1}{2} + \frac{1}{2} \cos 2x$$

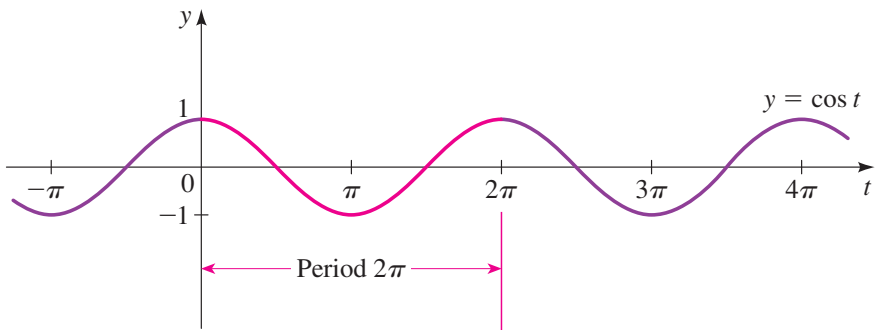
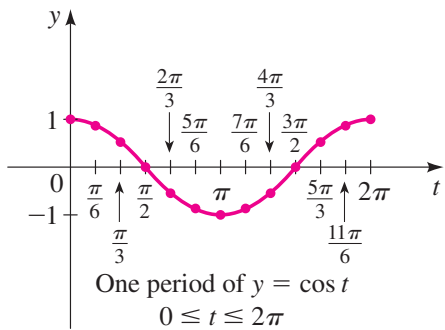
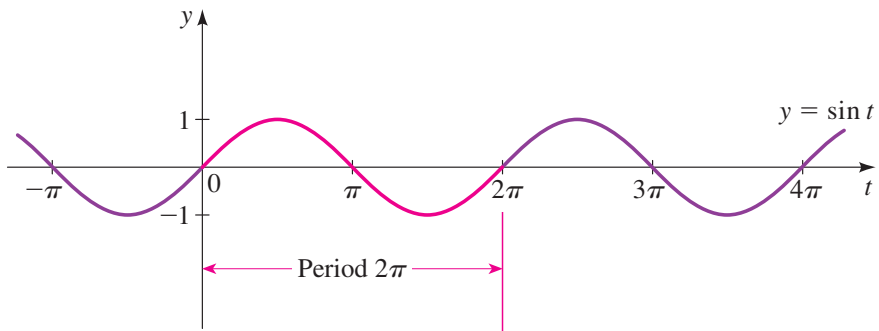
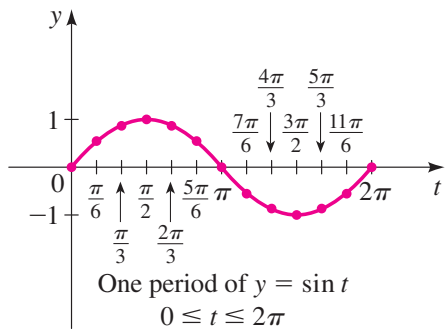
that is,

$$a_0 = 1, \quad a_1 = 0, \quad a_2 = \frac{1}{2}, \quad a_3 = a_4 = \dots = 0$$

and

$$b_1 = b_2 = b_3 = \dots = 0$$

Appendix I



Appendix II

We have

$$\begin{aligned}\int_{-2}^0 \cos \frac{n\pi x}{2} dx &= \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_{-2}^0 = \frac{2}{n\pi} \left(\sin \frac{n\pi(0)}{2} - \sin \frac{n\pi(-2)}{2} \right) \\ &= \frac{2}{n\pi} (\sin 0 - \sin(-n\pi)) \\ &= \frac{2}{n\pi} (0 - 0) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}\int_0^2 x \cos \frac{n\pi x}{2} dx &= \left[\begin{array}{l} x = u \quad \left| \quad \cos \frac{n\pi x}{2} dx = dv \\ dx = du \quad \left| \quad \frac{2}{n\pi} \sin \frac{n\pi x}{2} = v \end{array} \right. \right] = x \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^2 - \int_0^2 \frac{2}{n\pi} \sin \frac{n\pi x}{2} dx \\ &= \left(\frac{4}{n\pi} \sin n\pi - 0 \right) - \frac{2}{n\pi} \int_0^2 \sin \frac{n\pi x}{2} dx \\ &= -\frac{2}{n\pi} \int_0^2 \sin \frac{n\pi x}{2} dx \\ &= \left(-\frac{2}{n\pi} \right) \left(-\frac{2}{n\pi} \cos \frac{n\pi x}{2} \right) \Big|_0^2 \\ &= \frac{4}{(n\pi)^2} (\cos n\pi - \cos 0) \\ &= \frac{4}{(n\pi)^2} (\cos n\pi - 1)\end{aligned}$$

therefore

$$\begin{aligned}\frac{1}{2} \int_{-2}^2 f(x) \cos \frac{n\pi x}{2} dx &= \frac{1}{2} \int_{-2}^0 \cos \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \cos \frac{n\pi x}{2} dx \\ &= \frac{1}{2} \cdot 0 + \frac{1}{2} \cdot \frac{4}{(n\pi)^2} (\cos n\pi - 1) = \frac{2}{(n\pi)^2} (\cos n\pi - 1)\end{aligned}$$

We have

$$\begin{aligned}
 \int_{-2}^0 \sin \frac{n\pi x}{2} dx &= -\frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_{-2}^0 = -\frac{2}{n\pi} \left(\cos \frac{n\pi(0)}{2} - \cos \frac{n\pi(-2)}{2} \right) \\
 &= -\frac{2}{n\pi} (\cos 0 - \cos(-n\pi)) \\
 &= -\frac{2}{n\pi} (1 - \cos n\pi)
 \end{aligned}$$

and

$$\begin{aligned}
 \int_0^2 x \sin \frac{n\pi x}{2} dx &= \left[\begin{array}{l} x = u \quad \left| \quad \sin \frac{n\pi x}{2} dx = dv \\ dx = du \quad \left| \quad -\frac{2}{n\pi} \cos \frac{n\pi x}{2} = v \end{array} \right. \right] = -x \frac{2}{n\pi} \cos \frac{n\pi x}{2} \Big|_0^2 - \int_0^2 \left(-\frac{2}{n\pi} \right) \cos \frac{n\pi x}{2} dx \\
 &= \left(-\frac{4}{n\pi} \cos n\pi + 0 \right) + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} dx \\
 &= -\frac{4}{n\pi} \cos n\pi + \frac{2}{n\pi} \int_0^2 \cos \frac{n\pi x}{2} dx \\
 &= -\frac{4}{n\pi} \cos n\pi + \frac{2}{n\pi} \cdot \frac{2}{n\pi} \sin \frac{n\pi x}{2} \Big|_0^2 \\
 &= -\frac{4}{n\pi} \cos n\pi + \frac{4}{(n\pi)^2} (\sin n\pi - \sin 0) \\
 &= -\frac{4}{n\pi} \cos n\pi + \frac{4}{(n\pi)^2} (0 - 0) \\
 &= -\frac{4}{n\pi} \cos n\pi
 \end{aligned}$$

therefore

$$\begin{aligned}
 \frac{1}{2} \int_{-2}^2 f(x) \sin \frac{n\pi x}{2} dx &= \frac{1}{2} \int_{-2}^0 \sin \frac{n\pi x}{2} dx + \frac{1}{2} \int_0^2 x \sin \frac{n\pi x}{2} dx \\
 &= \frac{1}{2} \cdot \left(-\frac{2}{n\pi} (1 - \cos n\pi) \right) + \frac{1}{2} \cdot \left(-\frac{4}{n\pi} \cos n\pi \right) \\
 &= -\frac{1}{n\pi} (1 - \cos n\pi) - \frac{2}{n\pi} \cos n\pi \\
 &= -\frac{1}{n\pi} + \frac{1}{n\pi} \cos n\pi - \frac{2}{n\pi} \cos n\pi \\
 &= -\frac{1}{n\pi} - \frac{1}{n\pi} \cos n\pi \\
 &= -\frac{1}{n\pi} (1 + \cos n\pi)
 \end{aligned}$$