The Heat Equation; Separation of Variables

Consider the boundary-value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \qquad u(x,0) = f(x), \quad 0 < x < l; \quad u(0,t) = u(l,t) = 0 \tag{1}$$

Our goal is to find the solution $u(x,t)$ of (1). To this end, we will do the following:
(a) Find as many solutions $u_1(x,t), u_2(x,t), \ldots$ as we can of the boundary-value problem

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \qquad u(0,t) = u(l,t) = 0 \tag{2}$$

(b) Find the solution $u(x,t)$ of (1) by taking an appropriate linear combination of the functions $u_n(x,t)$, $n = 1, 2, \ldots$

Set

$$u(x,t) = X(x)T(t)$$

Computing

$$\frac{\partial u}{\partial t} = XT' \qquad \text{and} \qquad \frac{\partial^2 u}{\partial x^2} = X''T$$

we see that $u(x,t) = X(x)T(t)$ is a solution of the equation $u_t = \alpha^2 u_{xx}$ ($u_t = \partial u/\partial t$ and $u_{xx} = \partial^2 u/\partial x^2$) if

$$XT' = \alpha^2 X''T \tag{3}$$

Dividing both sides of (3) by $\alpha^2 XT$ gives

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} \tag{4}$$

Now, observe that the left-hand side of (4) is a function of x alone, while the right-hand side of (4) is a function of t alone. This implies that

$$\frac{X''}{X} = -\lambda \qquad \text{and} \qquad \frac{T'}{\alpha^2 T} = -\lambda \tag{5}$$

for some constant λ. (The only way that a function of x can equal a function of t is if both are constant. To convince yourself of this, let $f(x) = g(t)$ and fix t_0. Then, $f(x) = g(t_0)$ for all x, so that $f(x) =$ constant $= c_1$, and this immediately implies that $g(t)$ also equals c_1.) In addition, the boundary conditions

$$0 = u(0,t) = X(0)T(t)$$

and

$$0 = u(l,t) = X(l)T(t)$$

imply that $X(0) = 0$ and $X(l) = 0$ (otherwise, u must be identically zero). Thus, $u(x,t) = X(x)T(t)$ is a solution of (2) if

$$X'' + \lambda X = 0; \qquad X(0) = 0, \quad X(l) = 0 \tag{6}$$

and

$$T' + \lambda \alpha^2 T = 0 \tag{7}$$

At this point, the constant λ is arbitrary. However, we know from Example 1 of Section 5.1 that the boundary-value problem (6) has a nontrivial solution $X(x)$ only if

$$X(x) = X_n(x) = \sin\left(\frac{n\pi x}{l}\right) \qquad \text{for} \quad \lambda = \lambda_n = \frac{n^2\pi^2}{l^2}, \; n = 1, 2, \ldots$$

Equation (7), in turn, implies that

$$T(t) = T_n(t) = e^{-\lambda\alpha^2 t} = e^{-\alpha^2 n^2 \pi^2 t/l^2} \qquad \text{(see Appendix)}$$

(Actually, we should multiply both $X_n(x)$ and $T_n(t)$ by constants; however, we omit these constants here since we will soon be taking linear combinations of the functions $X_n(x)T_n(t)$.) Hence,

$$u_n(x,t) = \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t/l^2}$$

is a nontrivial solution of (2) for every positive integer n.

(b) Suppose that $f(x)$ is a finite linear combination of the functions $\sin\left(\frac{n\pi x}{l}\right)$; that is,

$$f(x) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right)$$

Then,

$$u(x,t) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right) e^{-\alpha^2 n^2 \pi^2 t/l^2}$$

is the desired solution of (1), since it is a linear combination of solutions of (2), and it satisfies the initial condition

$$u(x,0) = \sum_{n=1}^{N} c_n \sin\left(\frac{n\pi x}{l}\right) = f(x), \qquad 0 < x < l$$

EXAMPLE: At time $t = 0$, the temperature $u(x,0)$ in a thin copper rod ($\alpha^2 = 1.14$) of length one is

$$2\sin(3\pi x) + 5\sin(8\pi x), \qquad 0 \le x \le 1$$

The ends of the rod are packed in ice, so as to maintain them at 0°C. Find the temperature $u(x,t)$ in the rod at any time $t > 0$.

Solution: The temperature $u(x,t)$ satisfies the boundary-value problem and this implies that

$$\frac{\partial u}{\partial t} = 1.14\frac{\partial^2 u}{\partial x^2}; \qquad \begin{cases} u(x,0) = 2\sin(3\pi x) + 5\sin(8\pi x), & 0 < x < 1 \\ u(0,t) = u(1,t) = 0 \end{cases}$$

and this implies that

$$u(x,t) = 2\sin(3\pi x)e^{-9(1.14)\pi^2 t} + 5\sin(8\pi x)e^{-64(1.14)\pi^2 t}$$

Unfortunately, though, most functions $f(x)$ cannot be expanded as a finite linear combination of the functions $\sin\left(\frac{n\pi x}{l}\right)$, $n = 1, 2, \ldots$, on the interval $0 < x < l$. This leads us to ask the following question.

QUESTION: Can an arbitrary function $f(x)$ be written as an *infinite* linear combination of the functions $\sin\left(\frac{n\pi x}{l}\right)$, $n = 1, 2, \ldots$, on the interval $0 < x < l$? In other words, given an arbitrary function f, can we find constants $c_1, c_2, \ldots$, such that

$$f(x) = c_1 \sin\left(\frac{\pi x}{l}\right) + c_2 \sin\left(\frac{2\pi x}{l}\right) + \ldots = \sum_{n=1}^{\infty} c_n \sin\left(\frac{n\pi x}{l}\right); \qquad 0 < x < l?$$

Remarkably, the answer to this question is yes, as we show in Section 5.5.

Appendix

We have

$$T' + \lambda\alpha^2 T = 0$$

$$\frac{dT}{dt} + \lambda\alpha^2 T = 0$$

$$\frac{dT}{dt} = -\lambda\alpha^2 T$$

$$\frac{1}{T} dT = -\lambda\alpha^2 dt$$

$$\int \frac{1}{T} dT = -\lambda\alpha^2 \int dt$$

$$\ln|T| = -\lambda\alpha^2 t + C$$

If $C = 0$, we get

$$\ln|T| = -\lambda\alpha^2 t$$

hence

$$|T| = e^{-\lambda\alpha^2 t}$$

which gives

$$T = \pm e^{-\lambda\alpha^2 t}$$

In particular,

$$T = e^{-\lambda\alpha^2 t}$$

is a nontrivial solution of $T' + \lambda\alpha^2 T = 0$. Setting here

$$\lambda = \frac{n^2\pi^2}{l^2}$$

we obtain

$$T(t) = e^{-\alpha^2 n^2 \pi^2 t / l^2}$$