Introduction to Partial Differential Equations

Up to this point, the differential equations that we have studied have all been relations involving one or more functions of a single variable, and their derivatives. In this sense, these differential equations are *ordinary* differential equations. On the other hand, many important problems in applied mathematics give rise to *partial* differential equations. A partial differential equation is a relation involving one or more functions of *several* variables, and their partial derivatives. For example, the equation

$$\frac{\partial^3 u}{\partial x^3} + \left(\frac{\partial u}{\partial t}\right)^2 = \frac{\partial^2 u}{\partial x^2}$$

is a partial differential equation for the function $u(x, t)$, and the equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are a system of partial differential equations for the two functions $u(x, y)$ and $v(x, y)$ (for more examples, see the Appendix). The order of a partial differential equation is the order of the highest partial derivative that appears in the equation. For example, the order of the partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = 2\frac{\partial^2 u}{\partial x \partial t} + u$$

is two, since the order of the highest partial derivative that appears in this equation is two.

There are three classical partial differential equations of order two which appear quite often in applications, and which dominate the theory of partial differential equations. These equations are

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2} \tag{1}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \tag{2}$$

and

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{3}$$

Equation (1) is known as the heat equation, and it appears in the study of heat conduction and other diffusion processes. For example, consider a thin metal bar of length l whose surface is insulated. Let $u(x, t)$ denote the temperature in the bar at the point x at time t. This function satisfies the partial differential equation (1) for $0 < x < l$. The constant α^2 is known as the thermal diffusivity of the bar, and it depends solely on the material from which the bar is made.

Equation (2) is known as the wave equation, and it appears in the study of acoustic waves, water waves and electromagnetic waves. Some form of this equation, or a generalization of it, almost invariably arises in any mathematical analysis of phenomena involving the propagation of waves in a continuous medium. (We will gain some insight into why this is so in Section 5.7.) The wave equation also appears in the study of mechanical vibrations. Suppose, for example, that an elastic string of length l, such as a violin string or guy wire, is set in motion so that it vibrates in a vertical plane. Let $u(x, t)$ denote the vertical displacement of the string at the point x at time t (see Figure 1). If all damping effects, such as air resistance, are negligible, and if the amplitude of the motion is not too large, then $u(x, t)$ will satisfy the partial differential equation (2) on the interval $0 \le x \le l$. In this case, the constant c^2 is H/ρ, where H is the horizontal component of the tension in the string, and ρ is the mass per unit length of the string.

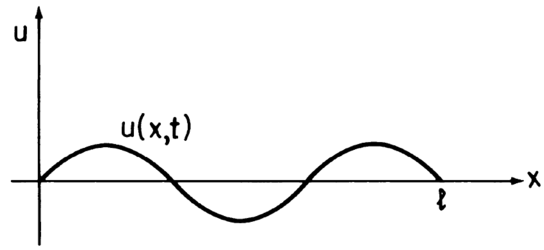

Figure 1

Equation (3) is known as Laplace's equation, and is the most famous of all partial differential equations. It arises in the study of such diverse applications as steady state heat flow, vibrating membranes, and electric and gravitational potentials. For this reason, Laplace's equation is often referred to as the potential equation.

In addition to the differential equation (1), (2), or (3), we will often impose initial and boundary conditions on the function u. These conditions will be dictated to us by the physical and biological problems themselves; they will be chosen so as to guarantee that our equation has a unique solution.

As a model case for the heat equation (1), we consider a thin metal bar of length l whose sides are insulated, and we let $u(x,t)$ denote the temperature in the bar at the point x at time t. In order to determine the temperature in the bar at any time t we need to know (i) the initial temperature distribution in the bar, and (ii) what is happening at the ends of the bar. Are they held at constant temperatures, say 0°C, or are they insulated, so that no heat can pass through them? (This latter condition implies that $u_x(0,t) = u_x(l,t) = 0$.) Thus, a "well posed" problem for diffusion processes is the heat equation (1), together with the initial condition

$$u(x,0) = f(x), \quad 0 < x < l$$

and the boundary conditions

$$u(0,t) = u(l,t) = 0 \quad \text{or} \quad u_x(0,t) = u_x(l,t) = 0$$

As a model case for the wave equation, we consider an elastic string of length l, whose ends are fixed, and which is set in motion in a vertical plane. In order to determine the position $u(x,t)$ of the string at any time t we need to know (i) the initial position of the string, and (ii) the initial velocity of the string. It is also implicit that $u(0,t) = u(l,t) = 0$. Thus, a well posed problem for wave propagation is the differential equation (2) together with the initial conditions $u(x,0) = f(x)$, $u_t(x,0) = g(x)$ and the boundary conditions $u(0,t) = u(l,t) = 0$.

The partial differential equation (3) does not contain the time t, so that we do not expect any "initial conditions" to be imposed here. In the problems that arise in applications, we are given u, or its normal derivative, on the boundary of a given region R, and we seek to determine $u(x,y)$ inside R. The problem of finding a solution of Laplace's equation which takes on given boundary values is known as a Dirichlet problem, while the problem of finding a solution of Laplace's equation whose normal derivative takes on given boundary values is known as a Neumann problem.

In Section 5.3 we will develop a very powerful method, known as the method of separation of variables, for solving the boundary-value problem (strictly speaking, we should say "initial boundary-value problem")

$$\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}; \qquad u(x,0) = f(x), \quad 0 < x < l; \quad u(0,t) = u(l,t) = 0$$

After developing the theory of Fourier series in Sections 5.4 and 5.5, we will show that the method of separation of variables can also be used to solve more general problems of heat conduction, and several important problems of wave propagation and potential theory.

Appendix

EXAMPLES:

1. (a) $\dfrac{\partial u}{\partial x} = c \quad \Longrightarrow \quad u(x,y) = g(y) + cx + d \quad \left(\text{Compare}: \dfrac{df}{dx} = c \quad \Longrightarrow \quad f(x) = cx + d\right)$

   (b) $\dfrac{\partial u}{\partial x} + \dfrac{\partial u}{\partial y} = c \quad \Longrightarrow \quad u(x,y) = g(y-x) + cx + d$

2. (a) $\dfrac{\partial^2 u}{\partial x^2} = c \quad \Longrightarrow \quad u(x,y) = \dfrac{1}{2}cx^2 + xg(y) + h(y) + d$

   (b) $\dfrac{\partial^2 u}{\partial y^2} = c \quad \Longrightarrow \quad u(x,y) = \dfrac{1}{2}cy^2 + yg(x) + h(x) + d$

   (c) $\dfrac{\partial^2 u}{\partial x \partial y} = c \quad \Longrightarrow \quad u(x,y) = cxy + g(x) + h(y) + d$

3. (a) $\dfrac{\partial^2 u}{\partial x^2} = cx \quad \Longrightarrow \quad u(x,y) = \dfrac{1}{6}cx^3 + xg(y) + h(y) + d$

   (b) $\dfrac{\partial^2 u}{\partial y^2} = cx \quad \Longrightarrow \quad u(x,y) = \dfrac{1}{2}cxy^2 + g(x)y + h(x) + d$

   (c) $\dfrac{\partial^2 u}{\partial x \partial y} = cx \quad \Longrightarrow \quad u(x,y) = \dfrac{1}{2}cx^2y + g(x) + h(y) + d$

4. $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial u}{\partial x} + u = c \quad \Longrightarrow \quad u(x,y) = e^{-x/2}\sin\left(\dfrac{1}{2}\sqrt{3}x\right)g(y) + e^{-x/2}\cos\left(\dfrac{1}{2}\sqrt{3}x\right)h(y) + c$

5. $\dfrac{\partial^2 u}{\partial x^2} + \left(\dfrac{\partial u}{\partial x}\right)^2 = 0 \quad \Longrightarrow \quad u(x,y) = \ln[xg(y) + h(y)] + c$

6. $\dfrac{\partial u}{\partial t} = \alpha^2 \dfrac{\partial^2 u}{\partial x^2}; \qquad u(x,0) = f(x), \quad 0 < x < l; \quad u(0,t) = u(l,t) = 0 \quad \Longrightarrow$

$$u(x,t) = \frac{2}{l}\sum_{n=1}^{\infty}\left[\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]\sin\left(\frac{n\pi x}{l}\right)e^{-\alpha^2 n^2 \pi^2 t/l^2}$$